

Spectrum analysis and optimal decay rates of the bipolar Vlasov-Poisson-Boltzmann equations

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Abstract

In the present paper, we consider the initial value problem for the bipolar Vlasov-Poisson-Boltzmann (bVPB) system and its corresponding modified Vlasov-Poisson-Boltzmann (mVPB). We give the spectrum analysis on the linearized bVPB and mVPB systems around their equilibrium state and show the optimal convergence rate of global solutions. It was showed that the electric field decays exponentially and the distribution function tends to the absolute Maxwellian at the optimal convergence rate $(1+t)^{-3/4}$ for the bVPB system, yet both the electric field and the distribution function converge to equilibrium state at the optimal rate $(1+t)^{-3/4}$ for the mVPB system.

Key words. Bipolar Vlasov-Poisson-Boltzmann system, Modified Vlasov-Poisson-Boltzmann, spectrum analysis, optimal time decay rates.

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Contents

1	Introduction	2
2	Main results	4
2.1	bVPB system	4
2.2	mVPB system	7
3	Analysis of spectra and semigroup for linear systems	10
3.1	Spectrum and resolvent of linear bVPB system	10
3.2	Exponential decay of semigroup for linear bVPB	14
3.3	Analysis of spectrum and semigroup for linear mVPB system	18
3.4	Optimal time-decay rates for linear mVPB	21
4	The nonlinear problem for bVPB system	24
4.1	Energy estimates	24
4.2	Convergence rates	30

5	The nonlinear problem for mVPB system	33
5.1	Energy estimates	33
5.2	Convergence rates	36

1 Introduction

The bipolar Vlasov-Poisson-Boltzmann (bVPB) system of two species can be used to model the time evolution of dilute charged particles (e.g., electrons and ions) in the absence of an external magnetic field [12]. In general, the bVPB system for two species of particles in the whole space take the form

$$\partial_t F_+ + v \cdot \nabla_x F_+ + \nabla_x \Phi \cdot \nabla_v F_+ = \mathcal{Q}(F_+, F_+) + \mathcal{Q}(F_+, F_-), \quad (1.1)$$

$$\partial_t F_- + v \cdot \nabla_x F_- - \nabla_x \Phi \cdot \nabla_v F_- = \mathcal{Q}(F_-, F_-) + \mathcal{Q}(F_-, F_+), \quad (1.2)$$

$$\Delta_x \Phi = \int_{\mathbb{R}^3} (F_+ - F_-) dv, \quad (1.3)$$

$$F_+(x, v, 0) = F_{+,0}(x, v), \quad F_-(x, v, 0) = F_{-,0}(x, v), \quad (1.4)$$

where $F_+ = F_+(x, v, t)$ and $F_- = F_-(x, v, t)$ are number density functions of ions and electrons, and $\Phi(x, t)$ denotes the electric potential, respectively. The collision integral $\mathcal{Q}(F, G)$ describes the interaction between particles due to binary collisions by

$$\mathcal{Q}(F, G) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot \omega| (F(v')G(v'_*) - F(v)G(v_*)) dv_* d\omega, \quad (1.5)$$

where

$$v' = v - [(v - v_*) \cdot \omega]\omega, \quad v'_* = v_* + [(v - v_*) \cdot \omega]\omega, \quad \omega \in \mathbb{S}^2.$$

Assume that the electron density is very rarefied and reaches a local equilibrium state with small electron mass compared with the ions, and that the collision $\mathcal{Q}(F_+, F_-)$ between the ions and electrons can be neglected, the equation (1.2) can be reduced to

$$v \cdot \nabla_x F_- - \nabla_x \Phi \cdot \nabla_v F_- = 0 \quad (1.6)$$

This together with the simple local Maxwellian distribution of electron leads to

$$F_- = \rho_-(x)M(v) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\Phi} e^{-\frac{|v|^2}{2}}$$

with the normalized Maxwellian $M(v)$ given by

$$M = M(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}.$$

For rigorous reduction of this type at the hydrodynamical scale, the readers can refer to [2]. Under the above reduction, we can obtain from bVPB system (1.1)–(1.3) the following modified Vlasov-Poisson-Boltzmann (mVPB) system:

$$F_t + v \cdot \nabla_x F + \nabla_x \Phi \cdot \nabla_v F = \mathcal{Q}(F, F), \quad (1.7)$$

$$\Delta_x \Phi = \int_{\mathbb{R}^3} F dv - e^{-\Phi}, \quad (1.8)$$

$$F(x, v, 0) = F_0(x, v), \quad (1.9)$$

where $F = F(x, v, t)$ is the distribution function of ions with $(x, v, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+$, $\Phi(x, t)$ is the electric potential, and $\mathcal{Q}(F, G)$ is the binary collision operator defined as (1.5).

In the case that the effect of electron is totally neglected, the bVPB system can be simplified to the standard unipolar Vlasov-Poisson-Boltzmann (VPB) model similar to (1.7)–(1.9) with the term $e^{-\Phi}$ replaced by a given function, a positive constant for instance.

There have been a lot of works on the existence and behavior of solutions to the Vlasov-Poisson-Boltzmann system. The global existence of renormalized solution for large initial data was proved in [13]. The first global existence result on classical solution in torus when the initial data is near a global Maxwellian was established in [7]. And the global existence of classical solution in \mathbb{R}^3 was given [18, 19] in the same setting. The case with general stationary background density function $\bar{\rho}(x)$ was studied in [4], and the perturbation of vacuum was investigated in [8, 5]. Recently, Li-Yang-Zhong [9] analyze the spectrum of the linearized VPB system and obtain the optimal decay rate of solutions to the nonlinear system near Maxwellian.

However, in contrast to the works on Boltzmann equation [6, 15, 16, 17] and VPB system [9], the spectrums of the linearized bVPB system and modified VPB system have not been given despite of its importance. On the other hand, an interesting phenomenon was shown recently in [3] on the time asymptotic behavior of the solutions which shows that the global classical solution of one species VPB system tends to the equilibrium at $(1+t)^{-\frac{1}{4}}$ in L^2 -norm. This is slower than the rate for the two species VPB system, that is, $(1+t)^{-\frac{3}{4}}$, obtained in [20]. Therefore, it is natural to investigate whether these rates are optimal.

The main purpose of the present paper is to investigate the spectrum and optimal time-convergence rates of global solutions to the linearized the bVPB system (1.1)–(1.4) in section 2.1 and the mVPB (1.7)–(1.9) in section 2.2 respectively. In particular, the main results established in this paper justifies how the electric field and the interplay interaction between ions and electrons influence the asymptotical behaviors of the global solution to the bVPB system (1.1)–(1.4) and the mVPB (1.7)–(1.9).

The rest of this paper will be organized as follows. The main results about the global existences and the optimal time-convergence rates of strong solution to bVPB system (1.1)–(1.4) and mVPB (1.7)–(1.9) are stated in Section 2. In Section 3, we analyze the spectrum of the bVPB system and mVPB system, and then establish the exponential time decay rates of the linearized bVPB and the algebraic time decay rates of the linearized mVPB equations in Sections 3.1–3.3. In Sections 4 and 5, we prove the optimal time decay rates of the global solution to the original nonlinear bVPB system and mVPB system respectively.

Notations: Define the Fourier transform of $f = f(x, v)$ by $\hat{f}(\xi, v) = \mathcal{F}f(\xi, v) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(x, v) e^{-ix \cdot \xi} dx$, where and throughout this paper we denote $i = \sqrt{-1}$.

Denote the weight function $w(v)$ by

$$w(v) = (1 + |v|^2)^{1/2}$$

and the Sobolev spaces H^N and H_w^N as

$$H^N = \{ f \in L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3) \mid \|f\|_{H^N} < \infty \}, \quad H_w^N = \{ f \in L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3) \mid \|f\|_{H_w^N} < \infty \}$$

equipped with the norms

$$\|f\|_{H^N} = \sum_{|\alpha|+|\beta| \leq N} \|\partial_x^\alpha \partial_v^\beta f\|_{L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)}, \quad \|f\|_{H_w^N} = \sum_{|\alpha|+|\beta| \leq N} \|w \partial_x^\alpha \partial_v^\beta f\|_{L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)}.$$

For $q \geq 1$, we also define

$$L^{2,q} = L^2(\mathbb{R}_v^3, L^q(\mathbb{R}_x^3)), \quad \|f\|_{L^{2,q}} = \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |f(x, v)|^q dx \right)^{2/q} dv \right)^{1/2}.$$

In the following, we denote by $\|\cdot\|_{L_{x,v}^2}$ and $\|\cdot\|_{L_{\xi,v}^2}$ the norms of the function spaces $L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$ and $L^2(\mathbb{R}_\xi^3 \times \mathbb{R}_v^3)$ respectively, and denote by $\|\cdot\|_{L_x^2}$, $\|\cdot\|_{L_\xi^2}$ and $\|\cdot\|_{L_v^2}$ the norms of the function spaces $L^2(\mathbb{R}_x^3)$, $L^2(\mathbb{R}_\xi^3)$ and $L^2(\mathbb{R}_v^3)$ respectively. For any integer $m \geq 1$, we denote by $\|\cdot\|_{H_x^m}$ and $\|\cdot\|_{L_v^2(H_x^m)}$ the norms in the spaces $H^m(\mathbb{R}_x^3)$ and $L^2(\mathbb{R}_v^3, H^m(\mathbb{R}_x^3))$ respectively.

2 Main results

2.1 bVPB system

First of all, we consider the Cauchy problem of the bVPB system (1.1)–(1.4) in the present paper. Define

$$F_1 =: F_+ + F_-, \quad F_2 =: F_+ - F_-.$$

Then Cauchy problem of the bVPB system (1.1)–(1.4) can be rewritten as

$$\partial_t F_1 + v \cdot \nabla_x F_1 + \nabla_x \Phi \cdot \nabla_v F_2 = \mathcal{Q}(F_1, F_1), \quad (2.10)$$

$$\partial_t F_2 + v \cdot \nabla_x F_2 + \nabla_x \Phi \cdot \nabla_v F_1 = \mathcal{Q}(F_2, F_1), \quad (2.11)$$

$$\Delta_x \Phi = \int_{\mathbb{R}^3} F_2 dv, \quad (2.12)$$

$$F_1(x, v, 0) = F_{1,0}(x, v) = F_{+,0} + F_{-,0}, \quad F_2(x, v, 0) = F_{2,0}(x, v) = F_{+,0} - F_{-,0}. \quad (2.13)$$

The bVPB system (2.10)–(2.12) has an equilibrium state $(F_1^*, F_2^*, \Phi^*) = (M(v), 0, 0)$. Define the perturbations $f_1(x, v, t)$ and $f_2(x, v, t)$ by

$$F_1 = M + \sqrt{M} f_1, \quad F_2 = \sqrt{M} f_2.$$

Then the bVPB system (2.10)–(2.12) for $f_1(x, v, t)$ and $f_2(x, v, t)$ is reformulated into

$$\partial_t f_1 + v \cdot \nabla_x f_1 - L f_1 = \frac{1}{2}(v \cdot \nabla_x \Phi) f_2 - \nabla_x \Phi \cdot \nabla_v f_2 + \Gamma(f_1, f_1), \quad (2.14)$$

$$\partial_t f_2 + v \cdot \nabla_x f_2 - v \sqrt{M} \cdot \nabla_x \Phi - L_1 f_2 = \frac{1}{2}(v \cdot \nabla_x \Phi) f_1 - \nabla_x \Phi \cdot \nabla_v f_1 + \Gamma(f_2, f_1), \quad (2.15)$$

$$\Delta_x \Phi = \int_{\mathbb{R}^3} f_2 \sqrt{M} dv, \quad (2.16)$$

$$f_1(x, v, 0) = f_{1,0}(x, v) = (F_{1,0} - M)M^{-\frac{1}{2}}, \quad f_2(x, v, 0) = f_{2,0}(x, v) = F_{2,0}M^{-\frac{1}{2}}, \quad (2.17)$$

where the operators $L f$, $L_1 f$ and $\Gamma(f, f)$ are defined by

$$L f = \frac{1}{\sqrt{M}} [\mathcal{Q}(M, \sqrt{M} f) + \mathcal{Q}(\sqrt{M} f, M)], \quad (2.18)$$

$$L_1 f = \frac{1}{\sqrt{M}} \mathcal{Q}(\sqrt{M} f, M), \quad (2.19)$$

$$\Gamma(f, g) = \frac{1}{\sqrt{M}} \mathcal{Q}(\sqrt{M} f, \sqrt{M} g). \quad (2.20)$$

The linearized collision operators L and L_1 can be written as [1, 21]

$$\begin{aligned} (L f)(v) &= (K f)(v) - \nu(v) f(v), \quad (L_1 f)(v) = (K_1 f)(v) - \nu(v) f(v), \\ \nu(v) &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot \omega| M_* d\omega dv_*, \\ (K f)(v) &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot \omega| (\sqrt{M'_*} f' + \sqrt{M'} f'_* - \sqrt{M} f_*) \sqrt{M_*} d\omega dv_* \\ &= \int_{\mathbb{R}^3} k(v, v_*) f(v_*) dv_*, \\ (K_1 f)(v) &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot \omega| \sqrt{M'_*} \sqrt{M_*} f' d\omega dv_* = \int_{\mathbb{R}^3} k_1(v, v_*) f(v_*) dv_*, \end{aligned}$$

where $\nu(v)$ is called the collision frequency, K and K_1 are self-adjoint compact operators on $L^2(\mathbb{R}_v^3)$ with real symmetric integral kernels $k(v, v_*)$ and $k_1(v, v_*)$. The nullspace of the operator L , denoted by N_0 , is a subspace spanned by the orthogonal basis $\{\chi_j, j = 0, 1, \dots, 4\}$ with

$$\chi_0 = \sqrt{M}, \quad \chi_j = v_j \sqrt{M} \quad (j = 1, 2, 3), \quad \chi_4 = \frac{(|v|^2 - 3)\sqrt{M}}{\sqrt{6}}, \quad (2.21)$$

and the nullspace of the operator L_1 , denoted by N_1 , is a subspace spanned by \sqrt{M} .

We denote $L^2(\mathbb{R}^3)$ be a Hilbert space of complex-value functions $f(v)$ on \mathbb{R}^3 with the inner product and the norm

$$(f, g) = \int_{\mathbb{R}^3} f(v) \overline{g(v)} dv, \quad \|f\| = \left(\int_{\mathbb{R}^3} |f(v)|^2 dv \right)^{1/2}.$$

Let P_0, P_d be the projection operators from $L^2(\mathbb{R}^3_v)$ to the subspace N_0, N_1 with

$$P_0 f = \sum_{i=0}^4 (f, \chi_i) \chi_i, \quad P_1 = I - P_0, \quad (2.22)$$

$$P_d f = (f, \sqrt{M}) \sqrt{M}, \quad P_r = I - P_d. \quad (2.23)$$

From the Boltzmann's H-theorem, the linearized collision operators L and L_1 are non-positive and moreover, L and L_1 are locally coercive in the sense that there is a constant $\mu > 0$ such that

$$(Lf, f) \leq -\mu \|P_1 f\|^2, \quad f \in D(L), \quad (2.24)$$

$$(L_1 f, f) \leq -\mu \|P_r f\|^2, \quad f \in D(L_1), \quad (2.25)$$

where $D(L)$ and $D(L_1)$ are the domains of L and L_1 given by

$$D(L) = D(L_1) = \{f \in L^2(\mathbb{R}^3) \mid \nu(v)f \in L^2(\mathbb{R}^3)\}.$$

In addition, for the hard sphere model, ν satisfies

$$\nu_0(1 + |v|) \leq \nu(v) \leq \nu_1(1 + |v|). \quad (2.26)$$

From the system (2.14)–(2.17) for (f_1, f_2) , we have the following decoupled linearized system for f_1 and f_2 :

$$\partial_t f_1 = E f_1, \quad f_1(x, v, 0) = f_{1,0}(x, v), \quad (2.27)$$

$$\partial_t f_2 = B f_2, \quad f_2(x, v, 0) = f_{2,0}(x, v), \quad (2.28)$$

where

$$E f_1 = L f_1 - (v \cdot \nabla_x) f_1, \quad (2.29)$$

$$B f_2 = L_1 f_2 - (v \cdot \nabla_x) f_2 - v \sqrt{M} \cdot \nabla_x (-\Delta_x)^{-1} \int_{\mathbb{R}^3} f_2 \sqrt{M} dv. \quad (2.30)$$

The equation (2.27) is the linearized Boltzmann equation, its spectrum analysis and the optimal decay rate of the solution has already been made for instance in [15, 22]. Therefore, we only need to investigate the spectrum analysis and the decay rate of the solution to the linearized Vlasov-Poisson-Boltzmann type equation (2.28). Indeed, take Fourier transform to (2.27)–(2.28) in x to get

$$\partial_t \hat{f}_1 = \hat{E}(\xi) \hat{f}_1, \quad (2.31)$$

$$\partial_t \hat{f}_2 = \hat{B}(\xi) \hat{f}_2, \quad (2.32)$$

where the operators $\hat{E}(\xi), \hat{B}(\xi)$ are defined for $\xi \neq 0$ by

$$\hat{E}(\xi) = L_1 - i(v \cdot \xi), \quad \hat{B}(\xi) = L_1 - i(v \cdot \xi) - \frac{i(v \cdot \xi)}{|\xi|^2} P_d.$$

Then, we have

Theorem 2.1. *Let $\sigma(\hat{B}(\xi))$ denotes the spectrum of operator $\hat{B}(\xi)$ to the linear equation (2.32). There exist a constant $a_1 > 0$ such that it holds for all $\xi \neq 0$ that*

$$\sigma(\hat{B}(\xi)) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda < -a_1\}. \quad (2.33)$$

Let $\sigma(\hat{E}(\xi))$ denotes the spectrum of operator $\hat{E}(\xi)$ to the linear equation (2.31). Then, for any $r_0 > 0$ there exists $\alpha = \alpha(r_0) > 0$ so that it holds for $|\xi| \geq r_0$ that

$$\sigma(\hat{E}(\xi)) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq -\alpha\}. \quad (2.34)$$

There exists a constant $r_0 > 0$ such that the spectrum $\sigma(\hat{E}(\xi))$ for $\xi = s\omega$ with $|s| \leq r_0$ and $\omega \in \mathbb{S}^2$ consists of five points $\{\mu_j(s), j = -1, 0, 1, 2, 3\}$ on the domain $\operatorname{Re} \lambda > -\mu/2$, which are C^∞ functions of s for $|s| \leq r_0$ and satisfy the following asymptotical expansion for $|s| \leq r_0$

$$\begin{cases} \mu_{\pm 1}(s) = \pm i\sqrt{\frac{5}{3}}s - b_{\pm 1}s^2 + o(s^2), & \overline{\mu_1(s)} = \mu_{-1}(s), \\ \mu_0(s) = -b_0s^2 + o(s^2), \\ \mu_2(s) = \mu_3(s) = -b_2s^2 + o(s^2), \end{cases} \quad (2.35)$$

with constants $b_j > 0$, $-1 \leq j \leq 2$.

With above spectrum analysis, we can obtain the global existence and the time-asymptotical behavior of unique solution to the Cauchy problem for the linear bVPB system (2.31)–(2.32) as follows.

Theorem 2.2. Assume that $f_{1,0} \in L_v^2(H_x^N) \cap L^{2,q}$ for $N \geq 1$ and $q \in [1, 2]$. Then there is a globally unique solution $f_1(x, v, t) = e^{tE} f_{1,0}(x, v)$ to the linearized Boltzmann equation (2.27), which satisfies for any $\alpha, \alpha' \in \mathbb{N}^3$ with $|\alpha| \leq N$, $\alpha' \leq \alpha$ and $m = |\alpha - \alpha'|$ that

$$\|(\partial_x^\alpha e^{tE} f_{1,0}, \chi_j)\|_{L_x^2} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{m}{2}}(\|\partial_x^\alpha f_0\|_{L_{x,v}^2} + \|\partial_x^{\alpha'} f_0\|_{L^{2,q}}), \quad j = 0, 1, 2, 3, 4, \quad (2.36)$$

$$\|P_1(\partial_x^\alpha e^{tE} f_{1,0})\|_{L_{x,v}^2} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{m+1}{2}}(\|\partial_x^\alpha f_0\|_{L_{x,v}^2} + \|\partial_x^{\alpha'} f_0\|_{L^{2,q}}). \quad (2.37)$$

In addition, assume that $f_{1,0} \in L^2(\mathbb{R}_v^3; H^N(\mathbb{R}_x^3) \cap L^1(\mathbb{R}_x^3))$ for $N \geq 1$ and there exist positive constants $d_0, d_1 > 0$ and a small constant $r_0 > 0$ so that the Fourier transform $\hat{f}_{1,0}(\xi, v)$ of the initial data $f_{1,0}(x, v)$ satisfies that $\inf_{|\xi| \leq r_0} |(\hat{f}_{1,0}, \sqrt{M})| \geq d_0$, $\inf_{|\xi| \leq r_0} |(\hat{f}_{1,0}, \chi_4)| \geq d_1 \sup_{|\xi| \leq r_0} |(\hat{f}_{1,0}, \sqrt{M})|$ and $\sup_{|\xi| \leq r_0} |(\hat{f}_{1,0}, v\sqrt{M})| = 0$. Then global solution $f(x, v, t) = e^{tE} f_{1,0}(x, v)$ satisfies for two positive constants $C_2 \geq C_1$ that

$$C_1(1+t)^{-\frac{3}{4}-\frac{k}{2}} \leq \|\nabla_x^k(e^{tE} f_{1,0}, \chi_j)\|_{L_x^2} \leq C_2(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad j = 0, 1, 2, 3, 4, \quad (2.38)$$

$$C_1(1+t)^{-\frac{5}{4}-\frac{k}{2}} \leq \|\nabla_x^k P_1(e^{tE} f_{1,0})\|_{L_{x,v}^2} \leq C_2(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \quad (2.39)$$

for $t > 0$ sufficiently large and $k \geq 0$.

Furthermore, if $f_{2,0} \in L_v^2(H_x^N) \cap L^{2,1}$ for $N \geq 1$, then the global solution $f_2(x, v, t) = e^{tB} f_{2,0}(x, v)$ to the linear Vlasov-Poisson-Boltzmann type equation (2.28) exists globally in time and satisfies for $t > 0$

$$\|\partial_x^\alpha f_2(t)\|_{L_{x,v}^2} + \|\partial_x^\alpha \nabla_x \Phi(t)\|_{L_x^2} \leq C e^{-\frac{1}{2}a_1 t} (\|\partial_x^\alpha f_{2,0}\|_{L_{x,v}^2} + \|f_{2,0}\|_{L^{2,1}}) \quad (2.40)$$

for $0 \leq |\alpha| \leq N$, where $\nabla_x \Phi(t) = \nabla_x \Delta_x^{-1}(e^{tB} f_{2,0}, \sqrt{M})$.

With the help of optimal time decay rates on the linearized bVPB (2.27)–(2.28) given by Theorem 2.2, we can obtain the optimal decay rates of the global solution to original bVPB system (1.1)–(1.3) as follows.

Theorem 2.3. Assume that $f_{\pm,0} = (F_{\pm,0} - \frac{1}{2}M)M^{-\frac{1}{2}} \in H_w^N \cap L^{2,1}$ for $N \geq 4$ and $\|f_{\pm,0}\|_{H_w^N \cap L^{2,1}} \leq \delta_0$ for a constant $\delta_0 > 0$ small enough. Then there exists a globally unique solution (F_{\pm}, Φ) with $F_{\pm}(x, v, t) = \frac{1}{2}M + \sqrt{M}f_{\pm}(x, v, t)$ to the bVPB system (1.1)–(1.3), which satisfies

$$\|\partial_x^k(f_+, f_-)(t)\|_{L_{x,v}^2} \leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad (2.41)$$

$$\|\partial_x^k \nabla_x \Phi(t)\|_{L_x^2} \leq C\delta_0 e^{-dt}, \quad (2.42)$$

and in particular

$$\|\partial_x^k(f_+(t), \chi_j)\|_{L_x^2} + \|\partial_x^k(f_-(t), \chi_j)\|_{L_x^2} \leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad (2.43)$$

$$\|\partial_x^k(P_1 f_+, P_1 f_-)(t)\|_{L_{x,v}^2} \leq C\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \quad (2.44)$$

$$\|(P_1 f_+, P_1 f_-)(t)\|_{H_w^N} + \|\nabla_x(P_0 f_+, P_0 f_-)(t)\|_{L_v^2(H_x^{N-1})} \leq C\delta_0(1+t)^{-\frac{5}{4}}, \quad (2.45)$$

for $j = 0, 1, 2, 3, 4$, $k = 0, 1$ and a constant $d > 0$.

Moreover, the recombination (f_1, f_2) with $f_1 =: f_+ + f_-$, $f_2 =: f_+ - f_-$ is the global solution to the system (2.14)–(2.17) satisfies

$$\|\partial_x^k(f_1(t), \chi_j)\|_{L_x^2} \leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad j = 0, 1, 2, 3, 4, \quad (2.46)$$

$$\|\partial_x^k P_1 f_1(t)\|_{L_{x,v}^2} \leq C\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \quad (2.47)$$

$$\|\partial_x^k f_2(t)\|_{L_{x,v}^2} + \|\partial_x^k \nabla_x \Phi(t)\|_{L_x^2} \leq C\delta_0 e^{-dt}, \quad (2.48)$$

$$\|(P_1 f_1, P_r f_2)(t)\|_{H_w^N} + \|\nabla_x(P_0 f_1, P_d f_2)(t)\|_{L_v^2(H_x^{N-1})} \leq C\delta_0(1+t)^{-\frac{5}{4}}, \quad (2.49)$$

for $k = 0, 1$.

Theorem 2.4. *Let the assumptions of Theorem 2.3 hold. Assume further that there exist positive constants $d_0, d_1 > 0$ and a small constant $r_0 > 0$ so that $\hat{f}_{\pm,0} = (\hat{F}_{\pm,0} - \frac{1}{2}M)M^{-\frac{1}{2}}$ satisfies that $\inf_{|\xi| \leq r_0} |(\hat{f}_{1,0}, \sqrt{M})| \geq d_0$, $\sup_{|\xi| \leq r_0} |(\hat{f}_{1,0}, \chi_j)| = 0$ ($j = 1, 2, 3$) and $\inf_{|\xi| \leq r_0} |(\hat{f}_{1,0}, \chi_4)| \geq d_1 \sup_{|\xi| \leq r_0} |(\hat{f}_{1,0}, \sqrt{M})|$ with $\hat{f}_{1,0} = \hat{f}_{+,0} + \hat{f}_{-,0}$. Then, the global solution (F_{\pm}, Φ) with $F_{\pm}(x, v, t) = \frac{1}{2}M + \sqrt{M}f_{\pm}(x, v, t)$ to the bVPB system (1.1)–(1.3) satisfies*

$$C_1\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}} \leq \|\nabla_x^k f_{\pm}(t)\|_{L_{x,v}^2} \leq C_2\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad (2.50)$$

$$\|\nabla_x^k \nabla_x \Phi(t)\|_{L_x^2} \leq C\delta_0 e^{-dt}, \quad (2.51)$$

and in particular

$$C_1\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}} \leq \|\nabla_x^k(f_{\pm}(t), \chi_j)\|_{L_x^2} \leq C_2\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad (2.52)$$

$$C_1\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}} \leq \|\nabla_x^k P_1 f_{\pm}(t)\|_{L_{x,v}^2} \leq C_2\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \quad (2.53)$$

for $t > 0$ large with two positive constants $C_2 > C_1$, $j = 0, 1, 2, 3, 4$, and $k = 0, 1$.

Moreover, the recombination $f_1 = f_+ + f_-$, $f_2 = f_+ - f_-$ to the system (2.14)–(2.17) satisfies

$$C_1\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}} \leq \|\nabla_x^k(f_1(t), \chi_j)\|_{L_x^2} \leq C_2\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad (2.54)$$

$$C_1\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}} \leq \|\nabla_x^k P_1 f_1(t)\|_{L_{x,v}^2} \leq C_2\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \quad (2.55)$$

$$C_1\delta_0(1+t)^{-\frac{3}{4}} \leq \|f_1(t)\|_{H_w^N} \leq C_2\delta_0(1+t)^{-\frac{3}{4}}, \quad (2.56)$$

$$\|\nabla_x^k f_2(t)\|_{L_{x,v}^2} + \|\partial_x^k \nabla_x \Phi(t)\|_{L_x^2} \leq C\delta_0 e^{-dt}, \quad (2.57)$$

for $t > 0$ large with two constants $C_2 > C_1$, $j = 0, 1, 2, 3, 4$, and $k = 0, 1$.

2.2 mVPB system

Next, we deal with the global existence and uniqueness of solution to the Cauchy problem for the mVPB system (1.7)–(1.9) and the optimal time-convergence rate of the global solutions. The mVPB system (1.7)–(1.8) has an equilibrium state $(F_*, \Phi_*) = (M, 0)$ with $M = M(v)$ being the normalized global Maxwellian defined above. Define the perturbation $f(x, v, t)$ of F near M by

$$f = (F - M)M^{-\frac{1}{2}},$$

then the modified Vlasov-Poisson-Boltzmann system (1.7)-(1.9) for $f(x, v, t)$ reads

$$\partial_t f + v \cdot \nabla_x f - v \sqrt{M} \cdot \nabla_x \Phi - Lf = \frac{1}{2}(v \cdot \nabla_x \Phi)f - \nabla_x \Phi \cdot \nabla_v f + \Gamma(f, f), \quad (2.58)$$

$$(I - \Delta_x)\Phi = - \int_{\mathbb{R}^3} f \sqrt{M} dv + (e^{-\Phi} + \Phi - 1), \quad (2.59)$$

$$f(x, v, 0) = f_0(x, v) =: (F_0 - M)M^{-1/2}, \quad (2.60)$$

where the operators Lf and $\Gamma(f, f)$ are defined by (2.18) and (2.20) respectively.

From the modified VPB system (2.58)–(2.60), we have the following the linearized mVPB equation

$$\partial_t f = B_m f, \quad t > 0, \quad (2.61)$$

$$f(x, v, 0) = f_0(x, v), \quad (x, v) \in \mathbb{R}_x^3 \times \mathbb{R}_v^3, \quad (2.62)$$

where the linear operator B_m is defined by

$$B_m f = Lf - v \cdot \nabla_x f - v \sqrt{M} \cdot \nabla_x (I - \Delta_x)^{-1} \left(\int_{\mathbb{R}^3} f \sqrt{M} dv \right).$$

Take the Fourier transform to (2.61) with respect to x to get

$$\partial_t \hat{f} = \hat{B}_m(\xi) \hat{f}, \quad (2.63)$$

where

$$\hat{B}_m(\xi) = L - i(v \cdot \xi) - \frac{i(v \cdot \xi)}{1 + |\xi|^2} P_d.$$

Then, we have the spectrum analysis of the operator $\hat{B}_m(\xi)$ and the time-decay rates of the global solution to the linearized mVPB system (2.61)–(2.62) and establish its optimal time-decay rates as follows.

Theorem 2.5. *Let $\sigma(\hat{B}_m(\xi))$ denotes the spectrum of operator $\hat{B}_m(\xi)$ to the linear equation (2.63) for all $\xi \in \mathbb{R}^3$. Then, for any $r_0 > 0$ there exists $\alpha = \alpha(r_0) > 0$ so that it holds for $|\xi| \geq r_0$ that*

$$\sigma(\hat{B}_m(\xi)) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq -\alpha\}.$$

There exists a constant $r_0 > 0$ so that the spectrum $\lambda \in \sigma(B_m(\xi)) \subset \mathbb{C}$ for $\xi = s\omega$ with $|s| \leq r_0$ and $\omega \in \mathbb{S}^2$ consists of five points $\{\lambda_j(s), j = -1, 0, 1, 2, 3\}$ on the domain $\operatorname{Re} \lambda > -\mu/2$, which are C^∞ functions of s for $|s| \leq r_0$ and satisfy the following asymptotical expansion for $|s| \leq r_0$

$$\begin{cases} \lambda_{\pm 1}(s) = \pm i 2 \sqrt{\frac{2}{3}} s - a_{\pm 1} s^2 + o(s^2), & \overline{\lambda_1(s)} = \lambda_{-1}(s), \\ \lambda_0(s) = -a_0 s^2 + o(s^2), \\ \lambda_2(s) = \lambda_3(s) = -a_2 s^2 + o(s^2), \end{cases} \quad (2.64)$$

with constants $a_j > 0$, $-1 \leq j \leq 2$, defined in Lemma 3.16.

With above spectrum analysis, we can obtain the global existence and the time-asymptotical behavior of unique solution to the Cauchy problem for the linear mVPB system (2.61)–(2.62) as follows.

Theorem 2.6. *Assume that $f_0 \in L^2(\mathbb{R}_x^3; H^N(\mathbb{R}_x^3) \cap L^q(\mathbb{R}_x^3))$ for $N \geq 1$ and $q \in [1, 2]$. Then there is a globally unique solution $f(x, v, t) = e^{tB_m} f_0(x, v)$ to the linearized mVPB system (2.61)–(2.62), which satisfies for any $\alpha, \alpha' \in \mathbb{N}^3$ with $|\alpha| \leq N$, $\alpha' \leq \alpha$ and $k = |\alpha - \alpha'|$ that*

$$\sum_{j=0}^4 \|\partial_x^\alpha (e^{tB_m} f_0, \chi_j)\|_{L_x^2} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}} (\|\partial_x^\alpha f_0\|_{L_{x,v}^2} + \|\partial_x^{\alpha'} f_0\|_{L^{2,q}}), \quad (2.65)$$

$$\|(I - \Delta_x)^{-1} (\partial_x^\alpha e^{tB_m} f_0, \sqrt{M})\|_{H_x^1} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}} (\|\partial_x^\alpha f_0\|_{L_{x,v}^2} + \|\partial_x^{\alpha'} f_0\|_{L^{2,q}}), \quad (2.66)$$

$$\|P_1(\partial_x^\alpha e^{tB^m} f_0)\|_{L_{x,v}^2} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1}{2}} (\|\partial_x^\alpha f_0\|_{L_{x,v}^2} + \|\partial_x^{\alpha'} f_0\|_{L^{2,q}}). \quad (2.67)$$

In addition, assume that $f_0 \in L^2(\mathbb{R}_v^3; H^N(\mathbb{R}_x^3) \cap L^1(\mathbb{R}_x^3))$ for $N \geq 1$ and there exist positive constants $d_0, d_1 > 0$ and a small constant $r_0 > 0$ so that the Fourier transform $\hat{f}_0(\xi, v)$ of the initial data $f_0(x, v)$ satisfies that $\inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_0)| \geq d_0$, $\inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_4)| \geq d_1 \sup_{|\xi| \leq r_0} |(\hat{f}_0, \chi_0)|$ and $\sup_{|\xi| \leq r_0} |(\hat{f}_0, v\sqrt{M})| = 0$. Then global solution $f(x, v, t) = e^{tB^m} f_0(x, v)$ satisfies for two positive constants $C_2 \geq C_1$ that

$$C_1(1+t)^{-\frac{3}{4}-\frac{k}{2}} \leq \|\nabla_x^k(e^{tB^m} f_0, \chi_j)\|_{L_x^2} \leq C_2(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad (2.68)$$

$$C_1(1+t)^{-\frac{3}{4}-\frac{k}{2}} \leq \|\nabla_x^k(I - \Delta_x)^{-1}(e^{tB^m} f_0, \sqrt{M})\|_{H_x^1} \leq C_2(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad (2.69)$$

$$C_1(1+t)^{-\frac{5}{4}-\frac{k}{2}} \leq \|\nabla_x^k P_1(e^{tB^m} f_0)\|_{L_{x,v}^2} \leq C_2(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \quad (2.70)$$

for $t > 0$ sufficiently large, $j = 0, 1, 2, 3, 4$, and $k \geq 0$.

Then, we state the results on the global existence and the optimal time-asymptotical behavior of unique solution to the Cauchy problem for the mVPB system (2.58)–(2.60) blow.

Theorem 2.7. Assume that $f_0 \in H_w^N \cap L^{2,1}$ for $N \geq 4$ and $\|f_0\|_{H_w^N \cap L^{2,1}} \leq \delta_0$ for a constant $\delta_0 > 0$ small enough. Then, there exists a globally unique strong solution $f = f(x, v, t)$ to the mVPB system (2.58)–(2.60) satisfying

$$\sum_{j=0}^4 \|\partial_x^k(f(t), \chi_j)\|_{L_x^2} + \|\partial_x^k \Phi(t)\|_{H_x^1} \leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad (2.71)$$

$$\|\partial_x^k P_1 f(t)\|_{L_{x,v}^2} \leq C\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \quad (2.72)$$

$$\|P_1 f(t)\|_{H_w^N} + \|\nabla_x P_0 f(t)\|_{L_v^2(H_x^{N-1})} \leq C\delta_0(1+t)^{-\frac{5}{4}}, \quad (2.73)$$

for $k = 0, 1$ and $t > 0$.

We shall prove that the above convergence rates are indeed optimal in the following sense.

Theorem 2.8. Let the assumptions of Theorem 2.7 hold. Assume further that there exist positive constants $d_0, d_1 > 0$ and a small constant $r_0 > 0$ so that the Fourier transform $\hat{f}_0(\xi, v)$ satisfies $\inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_0)| \geq d_0$, $\sup_{|\xi| \leq r_0} |(\hat{f}_0, \chi_j)| = 0$ ($j = 1, 2, 3$) and $\inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_4)| \geq d_1 \sup_{|\xi| \leq r_0} |(\hat{f}_0, \chi_0)|$. Then, the global solution f to the mVPB system (2.58)–(2.60) satisfies

$$C_1\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}} \leq \|\nabla_x^k(f(t), \chi_j)\|_{L_x^2} \leq C_2\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad (2.74)$$

$$C_1\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}} \leq \|\nabla_x^k \nabla_x \Phi(t)\|_{L_x^2} \leq C_2\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad (2.75)$$

$$C_1\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}} \leq \|\nabla_x^k P_1 f(t)\|_{L_{x,v}^2} \leq C_2\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \quad (2.76)$$

$$C_1\delta_0(1+t)^{-\frac{3}{4}} \leq \|f(t)\|_{H_w^N} \leq C_2\delta_0(1+t)^{-\frac{3}{4}}, \quad (2.77)$$

for $t > 0$ sufficiently large, two positive constants $C_2 \geq C_1$, $j = 0, 1, 2, 3, 4$, and $k = 0, 1$.

Remark 2.9 (Example). The initial data $f_0 = f_1(x, v)$ defined below satisfies the assumptions of Theorem 2.8

$$f_1(x, v) = d_0 e^{\frac{r_0^2}{2}} e^{-\frac{x^2}{2}} \chi_0 + d_1 d_0 e^{\frac{r_0^2}{2}} e^{-\frac{x^2}{2}} \chi_4.$$

for a small positive constant d_0 .

Remark 2.10. The conditions on initial data in above theorems can be applied to the VPB system and the Boltzmann equation to obtain corresponding the optimal time decay rate (refer to [22]). Although the global solution to the modified Vlasov-Poisson-Boltzmann (mVPB) equation system takes the same optimal time decay rate $(1+t)^{-3/4}$ as the Boltzmann equation, yet it observed that the hyperbolic waves of the mVPB system propagates at a faster speed due to the influence of electric field.

3 Analysis of spectra and semigroup for linear systems

3.1 Spectrum and resolvent of linear bVPB system

We investigate the spectrum analysis and the decay rate of the solution to the linearized Vlasov-Poisson-Boltzmann type equation (2.28). In the followings, we are concerned with the spectral analysis of the operator $\hat{B}(\xi)$ and optimal time-decay rate of solution to linear VPB type equation (2.32).

Introduce a weighted Hilbert space $L_\xi^2(\mathbb{R}^3)$ for $\xi \neq 0$ as

$$L_\xi^2(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3) \mid \|f\|_\xi = \sqrt{(f, f)_\xi} < \infty\},$$

with the inner product defined by

$$(f, g)_\xi = (f, g) + \frac{1}{|\xi|^2} (P_d f, P_d g).$$

Since P_d is a self-adjoint projection operator, it follows that $(P_d f, P_d g) = (P_d f, g) = (f, P_d g)$ and hence

$$(f, g)_\xi = (f, g + \frac{1}{|\xi|^2} P_d g) = (f + \frac{1}{|\xi|^2} P_d f, g). \quad (3.1)$$

By (3.1), we have for any $f, g \in L_\xi^2(\mathbb{R}_v^3) \cap D(\hat{B}(\xi))$,

$$(\hat{B}(\xi) f, g)_\xi = (\hat{B}(\xi) f, g + \frac{1}{|\xi|^2} P_d g) = (f, (L + i(v \cdot \xi) + \frac{i(v \cdot \xi)}{|\xi|^2} P_d) g) = (f, \hat{B}(-\xi) g)_\xi. \quad (3.2)$$

We can regard $\hat{B}(\xi)$ as a linear operator from the space $L_\xi^2(\mathbb{R}^3)$ to itself because

$$\|f\|^2 \leq \|f\|_\xi^2 \leq (1 + |\xi|^{-2}) \|f\|^2, \quad \xi \neq 0.$$

Similarly to the proofs of Lemmas 2.6–2.7 in [9], we have the following lemmas.

Lemma 3.1. *The operator $\hat{B}(\xi)$ generates a strongly continuous contraction semigroup on $L_\xi^2(\mathbb{R}^3)$, which satisfies*

$$\|e^{t\hat{B}(\xi)} f\|_\xi \leq \|f\|_\xi, \quad \text{for any } t > 0, f \in L_\xi^2(\mathbb{R}_v^3). \quad (3.3)$$

Lemma 3.2. *For each $\xi \neq 0$, the spectrum of $\hat{B}(\xi)$ on the domain $\text{Re} \lambda \geq -\nu_0 + \delta$ for any $\delta > 0$ consists of isolated eigenvalues $\{\lambda_j(\xi)\}$ with $\text{Re} \lambda_j(\xi) < 0$.*

Now denote by T a linear operator on $L^2(\mathbb{R}_v^3)$ or $L_\xi^2(\mathbb{R}_v^3)$, and we define the corresponding norms of T by

$$\|T\| = \sup_{\|f\|=1} \|Tf\|, \quad \|T\|_\xi = \sup_{\|f\|_\xi=1} \|Tf\|_\xi.$$

Obviously,

$$(1 + |\xi|^{-2})^{-1} \|T\| \leq \|T\|_\xi \leq (1 + |\xi|^{-2}) \|T\|. \quad (3.4)$$

First, we consider the spectrum and resolvent sets of $\hat{B}(\xi)$ at high frequency. To this end, we define

$$c(\xi) = -\nu(v) - i(v \cdot \xi), \quad (3.5)$$

and decompose $\hat{B}(\xi)$ into

$$\begin{aligned} \lambda - \hat{B}(\xi) &= \lambda - c(\xi) - K_1 + \frac{i(v \cdot \xi)}{|\xi|^2} P_d \\ &= (I - K_1(\lambda - c(\xi))^{-1} + \frac{i(v \cdot \xi)}{|\xi|^2} P_d(\lambda - c(\xi))^{-1})(\lambda - c(\xi)). \end{aligned} \quad (3.6)$$

Then, we have the estimates on the right hand terms of (3.6) as follows.

Lemma 3.3. *There exists a constant $C > 0$ so that it holds:*

1. *For any $\delta > 0$, we have*

$$\sup_{x \geq -\nu_0 + \delta, y \in \mathbb{R}} \|K_1(x + iy - c(\xi))^{-1}\| \leq C\delta^{-15/13}(1 + |\xi|)^{-2/13}, \quad (3.7)$$

2. *For any $\delta > 0$, $r_0 > 0$, there is a constant $y_0 = (2r_0)^{5/3}\delta^{-2/3} > 0$ such that if $|y| \geq y_0$, we have*

$$\sup_{x \geq -\nu_0 + \delta, |\xi| \leq r_0} \|K_1(x + iy - c(\xi))^{-1}\| \leq C\delta^{-7/5}(1 + |y|)^{-2/5}, \quad (3.8)$$

3. *For any $\delta > 0$, $r_0 > 0$, we have*

$$\sup_{x \geq -\nu_0 + \delta, y \in \mathbb{R}} \|(v \cdot \xi)|\xi|^{-2}P_d(x + iy - c(\xi))^{-1}\| \leq C\delta^{-1}|\xi|^{-1}, \quad (3.9)$$

$$\sup_{x \geq -\nu_0 + \delta, |\xi| \geq r_0} \|(v \cdot \xi)|\xi|^{-2}P_d(x + iy - c(\xi))^{-1}\| \leq C(r_0^{-1} + 1)(\delta^{-1} + 1)|y|^{-1}. \quad (3.10)$$

Proof. The proof of (3.9) and (3.10) can be found in Lemma 2.3 in [9]. Since K_1 satisfies the same properties as K (see [21]):

$$\int_{\mathbb{R}^3} |k_1(v, v_*)| dv_* \leq C(1 + |v|)^{-1}, \quad \int_{\mathbb{R}^3} |k_1(v, v_*)|^2 dv_* \leq C,$$

we can prove (3.7) and (3.8) by a same argument as Lemma 2.2.6 in [17]. \square

By Lemma 3.3 and a similar argument as Lemma 2.4 in [9], we have the spectral gap of the operator $\hat{B}(\xi)$ for high frequency.

Lemma 3.4. *Let $\lambda(\xi) \in \sigma(\hat{B}(\xi))$ be any eigenvalue of $\hat{B}(\xi)$ in the domain $\text{Re}\lambda \geq -\nu_0 + \delta$ with $\delta > 0$ being a constant. Then, for any $r_0 > 0$, there exists $\alpha(r_0) > 0$ so that $\text{Re}\lambda(\xi) \leq -\alpha(r_0)$ for all $|\xi| \geq r_0$.*

Then, we investigate the spectrum and resolvent sets of $\hat{B}(\xi)$ at low frequency. To this end, we decompose $\lambda - \hat{B}(\xi)$ as follows

$$\lambda - \hat{B}(\xi) = \lambda P_d + \lambda P_r - Q(\xi) + iP_d(v \cdot \xi)P_r + iP_r(v \cdot \xi)(1 + \frac{1}{|\xi|^2})P_d, \quad (3.11)$$

where

$$Q(\xi) = L_1 - iP_r(v \cdot \xi)P_r. \quad (3.12)$$

Lemma 3.5. *Let $\xi \neq 0$ and $Q(\xi)$ defined by (3.12). We have*

1. *If $\lambda \neq 0$, then*

$$\|\lambda^{-1}P_r(v \cdot \xi)(1 + \frac{1}{|\xi|^2})P_d\|_\xi \leq C(|\xi| + 1)|\lambda|^{-1}. \quad (3.13)$$

2. *If $\text{Re}\lambda > -\mu$, then the operator $\lambda P_r - Q(\xi)$ is invertible on N_1^\perp and satisfies*

$$\|(\lambda P_r - Q(\xi))^{-1}\| \leq (\text{Re}\lambda + \mu)^{-1}, \quad (3.14)$$

$$\|P_d(v \cdot \xi)P_r(\lambda P_r - Q(\xi))^{-1}P_r\|_\xi \leq C(1 + |\lambda|)^{-1}[(\text{Re}\lambda + \mu)^{-1} + 1](1 + |\xi|)^2. \quad (3.15)$$

Proof. Since

$$\|\lambda^{-1}P_r(v \cdot \xi)(1 + \frac{1}{|\xi|^2})P_d f\|_\xi \leq C|\lambda|^{-1}(|\xi| + \frac{1}{|\xi|})\|P_d f\| \leq C|\lambda|^{-1}(|\xi| + 1)\|f\|_\xi,$$

we prove (3.13).

Then, we show that for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -\mu$, the operator $\lambda P_r - Q(\xi) = \lambda P_r - L_1 + i P_r(v \cdot \xi) P_r$ is invertible from N_1^\perp to itself. Indeed, by (2.25), we obtain for any $f \in N_1^\perp \cap D(L_1)$ that

$$\operatorname{Re}([\lambda P_r - L_1 + i P_r(v \cdot \xi) P_r]f, f) = \operatorname{Re}\lambda(f, f) - (L_1 f, f) \geq (\mu + \operatorname{Re}\lambda)\|f\|^2, \quad (3.16)$$

which implies that the operator $\lambda P_r - Q(\xi)$ is an one-to-one map from N_1^\perp to itself so long as $\operatorname{Re}\lambda > -\mu$. The estimate (3.14) follows directly from (3.16).

By (3.14) and $\|P_d(v \cdot \xi) P_r f\|_\xi \leq C(|\xi| + 1)\|P_r f\|$, we have

$$\|P_d(v \cdot \xi) P_r (\lambda P_r - Q(\xi))^{-1} P_r f\|_\xi \leq C(|\xi| + 1)(\operatorname{Re}\lambda + \mu)^{-1}\|f\|. \quad (3.17)$$

Meanwhile, we can decompose the operator $P_d(v \cdot \xi) P_r (\lambda P_r - Q(\xi))^{-1} P_r$ as

$$P_d(v \cdot \xi) P_r (\lambda P_r - Q(\xi))^{-1} P_r = \frac{1}{\lambda} P_d(v \cdot \xi) P_r + \frac{1}{\lambda} P_d(v \cdot \xi) P_r Q(\xi) (\lambda P_r - Q(\xi))^{-1} P_r.$$

This together with (3.14) and the fact $\|P_d(v \cdot \xi) P_r Q(\xi)\| \leq C(1 + |\xi|)^2$ give

$$\|P_d(v \cdot \xi) P_r (\lambda P_r - Q(\xi))^{-1} P_r f\|_\xi \leq C|\lambda|^{-1}[(\operatorname{Re}\lambda + \mu)^{-1} + 1](1 + |\xi|)^2\|f\|. \quad (3.18)$$

The combination of the two cases (3.17) and (3.18) yields (3.15). \square

Consider the eigenvalue problem

$$\lambda f = (L_1 - i(v \cdot \xi))f - \frac{i\sqrt{M}(v \cdot \xi)}{|\xi|^2} \int_{\mathbb{R}^3} f \sqrt{M} dv. \quad (3.19)$$

We shall prove that $\hat{B}(\xi)$ has a spectral gap when $|\xi|$ is sufficiently small. For convenience, we shall use the parametrization $\xi = s\omega$ where $s \in \mathbb{R}^1$, $\omega \in \mathbb{S}^2$.

Let f be the eigenfunction of (3.19), we rewrite f in the form $f = f_0 + f_1$, where $f_0 = P_d f = C_0 \sqrt{M}$ and $f_1 = (I - P_d)f = P_r f$. The eigenvalue problem (3.19) can be decomposed into

$$\lambda f_0 = -P_d[i(v \cdot \xi)(f_0 + f_1)], \quad (3.20)$$

$$\lambda f_1 = L_1 f_1 - P_r[i(v \cdot \xi)(f_0 + f_1)] - \frac{i(v \cdot \xi)}{|\xi|^2} f_0. \quad (3.21)$$

From Lemma 3.5 and (3.21), we obtain that for any $\operatorname{Re}\lambda > -\mu$

$$f_1 = i[L_1 - \lambda P_r - i P_r(v \cdot \xi) P_r]^{-1} P_r((v \cdot \xi) f_0 + \frac{(v \cdot \xi)}{|\xi|^2} f_0). \quad (3.22)$$

Substituting (3.22) into (3.20) and taking inner product the resulted equation with \sqrt{M} gives

$$\lambda C_0 = (1 + \frac{1}{|\xi|^2})(R(\lambda, \xi)(v \cdot \xi)\sqrt{M}, (v \cdot \xi)\sqrt{M})C_0. \quad (3.23)$$

where $R(\lambda, \xi) = [L_1 - \lambda P_r - i P_r(v \cdot \xi) P_r]^{-1}$.

By changing variable $(v \cdot \xi) \rightarrow s v_1$ and using the rotational invariance of the operator L_1 , we have the following transformation.

Lemma 3.6. *Let $e_1 = (1, 0, 0)$, $\xi = s\omega$ with $s \in \mathbb{R}, \omega \in \mathbb{S}^2$. Then*

$$(R(\lambda, \xi)(v \cdot \xi)\sqrt{M}, (v \cdot \xi)\sqrt{M}) = s^2(R(\lambda, s e_1)(v_1 \sqrt{M}), v_1 \sqrt{M}). \quad (3.24)$$

With the help of (3.24), we rewrite (3.23) in the form

$$\lambda C_0 = (1 + s^2)(R(\lambda, s e_1)\chi_1, \chi_1)C_0. \quad (3.25)$$

Denote

$$D(\lambda, s) = (1 + s^2)(R(\lambda, s e_1)\chi_1, \chi_1). \quad (3.26)$$

Lemma 3.7. *There exists a small constant $r_0 > 0$ such that the equation $\lambda = D(\lambda, s)$ has no solution for $\operatorname{Re} \lambda \geq -\frac{1}{4}a_0$ and $|s| \leq r_0$, where*

$$a_0 = \min\{\mu, -(L_1^{-1}\chi_1, \chi_1)\} > 0.$$

Proof. We prove this lemma by three steps. First, we prove

$$x \neq D(x, 0) := D_0(x) \quad \text{for } x > -a_0. \quad (3.27)$$

Indeed, since $D'_0(x) = \|(L_1 - xP_r)^{-1}\chi_1\|^2 > 0$, we have $D_0(x) < D_0(0) = (L_1^{-1}\chi_1, \chi_1) < -a_0$ for $-\mu < x < 0$. Thus (3.27) holds for $-a_0 < x < 0$. For $x \geq 0$, since $D_0(x) = (L_1g, g) - x\|g\|^2 < 0$ with $g = (L_1 - xP_r)^{-1}\chi_1 \in N_1^\perp$, it follows that (3.27) holds for $x \geq 0$. Thus we obtain (3.27).

Second, we show

$$\lambda \neq D(\lambda, 0) := D_0(\lambda) \quad \text{for } \operatorname{Re} \lambda > -\frac{1}{2}a_0. \quad (3.28)$$

In the case of $\operatorname{Im} \lambda = 0$, (3.28) follows from (3.27). Assume that $\lambda = x + iy$ with $y \neq 0$. Note that

$$\operatorname{Re} D_0(\lambda) = (L_1h, h) - x(h, h), \quad \operatorname{Im} D_0(\lambda) = y\|h\|^2,$$

where $h = (L_1 - \lambda P_r)^{-1}\chi_1 \in N_1^\perp$. If there is a $\lambda = x + iy$ with $x > -\frac{1}{2}a_0$ such that $\lambda = D_0(\lambda)$, then

$$x = (L_1h, h) - x(h, h), \quad (3.29)$$

$$y = y\|h\|^2. \quad (3.30)$$

By (3.30) and $y \neq 0$, we have $\|h\| = 1$. This together with (3.29) gives $2x = (L_1h, h) \leq -\mu\|h\|^2 = -\mu$, which contradicts to $2x > -a_0 \geq -\mu$. Thus we obtain (3.28).

Finally, by (3.14) we have $\lim_{|\lambda| \rightarrow \infty} |\lambda - D_0(\lambda)| = \infty$ for $\operatorname{Re} \lambda > -\mu$. This together with (3.28) and the continuity of $D_0(\lambda)$ imply that there is a constant $\delta_0 > 0$ so that $|\lambda - D_0(\lambda)| > \delta_0$ for $\operatorname{Re} \lambda \geq -\frac{1}{4}a_0$. Since

$$|D(\lambda, s) - D(\lambda, 0)| \leq |[R(\lambda, se_1) - R(\lambda, 0)]\chi_1, \chi_1| + s^2|(R(\lambda, se_1)\chi_1, \chi_1)| \leq C(s + s^2),$$

we obtain

$$|\lambda - D(\lambda, s)| \geq |\lambda - D(\lambda, 0)| - |D(\lambda, 0) - D(\lambda, s)| > 0, \quad \operatorname{Re} \lambda \geq -\frac{1}{4}a_0, \quad |s| \leq r_0,$$

for $r_0 > 0$ small enough. This completes the proof of lemma. \square

Lemma 3.8. *It holds that*

1. *There is a constant $a_1 > 0$ such that it holds for all $\xi \neq 0$ that*

$$\sigma(\hat{B}(\xi)) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda < -a_1\}. \quad (3.31)$$

2. *For any $\delta > 0$ and all $\xi \neq 0$, there exists $y_1(\delta) > 0$ such that*

$$\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq -\mu + \delta, |\operatorname{Im} \lambda| \geq y_1\} \cup \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq -a_1\} \subset \rho(\hat{B}(\xi)). \quad (3.32)$$

Proof. By Lemma 3.7 and Lemma 3.4, we can choose $0 < a_1 < \min\{\alpha(r_0), \frac{1}{3}a_0\}$ with $r_0, a_0 > 0$ given by Lemma 3.7 and $\alpha(r_0) > 0$ given by Lemma 3.4 so that if $\lambda(\xi) \in \sigma(\hat{B}(\xi))$, then $\operatorname{Re} \lambda(\xi) < -a_1$ for all $\xi \neq 0$. This proves (3.31). By (3.7) and (3.9), there exists $r_1 > 0$ large such that if $\operatorname{Re} \lambda \geq -\nu_0 + \delta$ and $|\xi| \geq r_1$, then

$$\|K_1(\lambda - c(\xi))^{-1}\| \leq 1/4, \quad \|(v \cdot \xi)|\xi|^{-2}P_d(\lambda - c(\xi))^{-1}\| \leq 1/4. \quad (3.33)$$

This implies that the operator $I + K_1(\lambda - c(\xi))^{-1} + i(v \cdot \xi)|\xi|^{-2}P_d(\lambda - c(\xi))^{-1}$ is invertible on $L_\xi^2(\mathbb{R}_v^3)$, which together with (3.6) yields that $(\lambda - \hat{B}(\xi))$ is also invertible on $L_\xi^2(\mathbb{R}_v^3)$ and satisfies

$$(\lambda - \hat{B}(\xi))^{-1} = (\lambda - c(\xi))^{-1}(I - K(\lambda - c(\xi))^{-1} + \frac{i(v \cdot \xi)}{|\xi|^2}P_d(\lambda - c(\xi))^{-1})^{-1}. \quad (3.34)$$

Thus $\rho(\hat{B}(\xi)) \supset \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq -\nu_0 + \delta_1\}$ for $|\xi| > r_1$.

By Lemmas 3.5, we have for $\operatorname{Re} \lambda > -\mu$ and $\lambda \neq 0$ that the operator $\lambda P_d + \lambda P_r - Q(\xi)$ is invertible on $L_\xi^2(\mathbb{R}_v^3)$ and it satisfies

$$(\lambda P_d + \lambda P_r - Q(\xi))^{-1} = \lambda^{-1} P_d + (\lambda P_r - Q(\xi))^{-1} P_r,$$

because the operator λP_d is orthogonal to $\lambda P_r - Q(\xi)$. Therefore, we can re-write (3.11) as

$$\begin{aligned} \lambda - \hat{B}(\xi) &= (I + Y_1(\lambda, \xi))(\lambda P_d + \lambda P_r - Q(\xi)), \\ Y_1(\lambda, \xi) &=: i P_d(v \cdot \xi) P_r (\lambda P_r - Q(\xi))^{-1} P_r + i \lambda^{-1} P_r(v \cdot \xi) \left(1 + \frac{1}{|\xi|^2}\right) P_d. \end{aligned} \quad (3.35)$$

For the case $|\xi| \leq r_1$, by (3.13) and (3.15) we can choose $y_1 = y_1(\delta) > 0$ such that it holds for $\operatorname{Re} \lambda \geq -\mu + \delta$ and $|\operatorname{Im} \lambda| \geq y_1$ that

$$\|\lambda^{-1} P_r(v \cdot \xi) \left(1 + \frac{1}{|\xi|^2}\right) P_d\|_\xi \leq \frac{1}{4}, \quad \|P_d(v \cdot \xi) P_r (\lambda P_r - Q(\xi))^{-1} P_r\|_\xi \leq \frac{1}{4}. \quad (3.36)$$

This implies that the operator $I + Y_1(\lambda, \xi)$ is invertible on $L_\xi^2(\mathbb{R}_v^3)$ and thus $\lambda - \hat{B}(\xi)$ is invertible on $L_\xi^2(\mathbb{R}_v^3)$ and satisfies

$$(\lambda - \hat{B}(\xi))^{-1} = [\lambda^{-1} P_d + (\lambda P_r - Q(\xi))^{-1} P_r] (I + Y_1(\lambda, \xi))^{-1}. \quad (3.37)$$

Therefore, $\rho(\hat{B}(\xi)) \supset \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq -\mu + \delta_1, |\operatorname{Im} \lambda| \geq y_1\}$ for $|\xi| \leq r_1$. This and (3.31) lead to (3.32). \square

3.2 Exponential decay of semigroup for linear bVPB

Lemma 3.9. *Let $Y_1(\lambda, \xi)$ be defined by (3.35) and let $r_0 > 0$ be given by Lemma 3.7 and $a_1 > 0$ be given by Lemma 3.8. If $-a_1 < \operatorname{Re} \lambda < 0$, then the operator $I + Y_1(\lambda, \xi)$ is invertible on $L_\xi^2(\mathbb{R}_v^3)$ and satisfies*

$$\sup_{0 < |\xi| < r_0, \operatorname{Im} \lambda \in \mathbb{R}} \|(I + Y_1(\lambda, \xi))^{-1}\|_\xi \leq C. \quad (3.38)$$

Proof. Let $\lambda = x + iy$. By (3.13) and (3.15), there exists $R > 0$ large enough such that if $|y| \geq R$, then

$$\|\lambda^{-1} P_r(v \cdot \xi) \left(1 + \frac{1}{|\xi|^2}\right) P_d\|_\xi \leq \frac{1}{4}, \quad \|P_d(v \cdot \xi) P_r (\lambda P_r - Q(\xi))^{-1} P_r\|_\xi \leq \frac{1}{4},$$

which yields

$$\|(I + Y_1(\lambda, \xi))^{-1}\|_\xi \leq 2.$$

Thus it is sufficient to prove (3.38) for $|y| \leq R$. We make use of the argument of contradiction. Indeed, if (3.38) does not hold for $|y| \leq R$, namely, there are subsequences $\xi_n, \lambda_n = x + iy_n$ with $|\xi_n| \leq r_0, |y_n| \leq R$, and f_n, g_n with $\|f_n\|_{\xi_n} \rightarrow 0$ ($n \rightarrow \infty$), $\|g_n\|_{\xi_n} = 1$ such that

$$(I + Y_1(\lambda_n, \xi_n))^{-1} f_n = g_n.$$

This gives

$$f_n = g_n + i P_d(v \cdot \xi_n) P_r (\lambda_n P_r - Q(\xi_n))^{-1} P_r g_n + i \lambda_n^{-1} P_r(v \cdot \xi_n) \left(1 + \frac{1}{|\xi_n|^2}\right) P_d g_n,$$

and then

$$P_d f_n = P_d g_n + i P_d(v \cdot \xi_n) P_r (\lambda_n P_r - Q(\xi_n))^{-1} P_r g_n, \quad (3.39)$$

$$P_r f_n = P_r g_n + i \lambda_n^{-1} P_r(v \cdot \xi_n) \left(1 + \frac{1}{|\xi_n|^2}\right) P_d g_n. \quad (3.40)$$

Substituting (3.40) into (3.39), we obtain

$$\begin{aligned} & P_d g_n - P_d f_n + i P_d (v \cdot \xi_n) P_r (\lambda_n P_r - Q(\xi_n))^{-1} P_r f_n \\ & + \lambda_n^{-1} P_d (v \cdot \xi_n) P_r (\lambda_n P_r - Q(\xi_n))^{-1} P_r (v \cdot \xi_n) (1 + \frac{1}{|\xi_n|^2}) P_d g_n = 0. \end{aligned} \quad (3.41)$$

Since $\|f_n\|_{\xi_n} \rightarrow 0$ as $n \rightarrow \infty$, it follows from (3.41) and (3.15) that

$$\lim_{n \rightarrow \infty} \|P_d g_n + \lambda_n^{-1} P_d (v \cdot \xi_n) P_r (\lambda_n P_r - Q(\xi_n))^{-1} P_r (v \cdot \xi_n) (1 + \frac{1}{|\xi_n|^2}) P_d g_n\|_{\xi_n} = 0.$$

Let $P_d g_n = C_n \sqrt{M}$. We obtain

$$\lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{|\xi_n|^2}} \frac{|C_n|}{|\lambda_n|} |\lambda_n + (|\xi_n|^2 + 1)((\lambda_n P_r - Q(|\xi_n| e_1))^{-1} (v_1 \sqrt{M}), v_1 \sqrt{M})| = 0. \quad (3.42)$$

Since $\sqrt{1 + \frac{1}{|\xi_n|^2}} |C_n| \leq 1$, $|\xi_n| \leq r_0$, $|y_n| \leq R$, there is a subsequence $(\xi_{n_j}, \lambda_{n_j}, C_{n_j})$ such that $\sqrt{1 + \frac{1}{|\xi_{n_j}|^2}} C_{n_j} \rightarrow A_0$, $\xi_{n_j} \rightarrow \xi_0$, $\lambda_{n_j} \rightarrow \tilde{\lambda} = x + iy \neq 0$. Thus

$$\frac{|A_0|}{|\tilde{\lambda}|} |\tilde{\lambda} + (|\xi_0|^2 + 1)((\tilde{\lambda} P_r - Q(|\xi_0| e_1))^{-1} (v_1 \sqrt{M}), v_1 \sqrt{M})| = 0. \quad (3.43)$$

It is easy to verify that $A_0 \neq 0$. Indeed, if not, we have $\lim_{j \rightarrow \infty} \|P_r (v \cdot \xi_{n_j}) (1 + |\xi_{n_j}|^{-2}) P_d g_{n_j}\| = 0$ and then $\lim_{j \rightarrow \infty} \|P_r g_n\| = 0$ due to (3.40). Thus $\lim_{j \rightarrow \infty} \|g_{n_j}\|_{\xi_{n_j}} = 0$, which contradicts to $\|g_n\|_{\xi_n} = 1$. Therefore, from (3.43) we have

$$\tilde{\lambda} = D(\tilde{\lambda}, |\xi_0|), \quad |\xi_0| \leq r_0, \quad -a_1 < \operatorname{Re} \tilde{\lambda} < 0.$$

This implies that $\tilde{\lambda}$ is an eigenvalue of $\hat{B}(\xi_0)$ with $\operatorname{Re} \tilde{\lambda} > -a_1$, which contradicts to Lemma 3.8. \square

Lemma 3.10. *The operator $Q(\xi) = L_1 - i P_r (v \cdot \xi) P_r$ generates a strongly continuous contraction semigroup on N_1^\perp , which satisfies for any $t > 0$ and $f \in N_1^\perp \cap L^2(\mathbb{R}_v^3)$ that*

$$\|e^{tQ(\xi)} f\| \leq e^{-\mu t} \|f\|. \quad (3.44)$$

In addition, for any $x > -\mu$ and $f \in N_1^\perp \cap L^2(\mathbb{R}_v^3)$ it holds that

$$\int_{-\infty}^{+\infty} \|(x + iy) P_r - Q(\xi)\|^{-1} f\|^2 dy \leq \pi (x + \mu)^{-1} \|f\|^2. \quad (3.45)$$

Proof. Since $Q(\xi)$ and $Q(\xi)^* = Q(-\xi)$ are dissipative operators satisfying (3.16), we can prove (3.44) and (3.45) by applying a similar argument as Lemma 3.1 in [9]. \square

Theorem 3.11. *The semigroup $e^{t\hat{B}(\xi)}$ satisfies*

$$\|e^{t\hat{B}(\xi)} f\|_\xi \leq C e^{-\frac{1}{2} a_1 t} \|f\|_\xi, \quad f \in L_\xi^2(\mathbb{R}_v^3), \quad (3.46)$$

where $a_1 > 0$ is a constant given by Lemma 3.8.

Proof. Since $D(\hat{B}(\xi)^2)$ is dense in $L_\xi^2(\mathbb{R}_v^3)$ by Theorem 2.7 in p.6 of [14], we only need to prove (3.46) for $f \in D(\hat{B}(\xi)^2)$. By Corollary 7.5 in p.29 of [14], the semigroup $e^{t\hat{B}(\xi)}$ can be represented by

$$e^{t\hat{B}(\xi)} f = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} e^{\lambda t} (\lambda - \hat{B}(\xi))^{-1} f d\lambda, \quad f \in D(\hat{B}(\xi)^2), \quad \kappa > 0. \quad (3.47)$$

By (3.37), we rewrite $(\lambda - \hat{B}(\xi))^{-1}$ for $|\xi| \leq r_0$ as

$$(\lambda - \hat{B}(\xi))^{-1} = \lambda^{-1} P_d + (\lambda P_r - Q(\xi))^{-1} P_r - Z_1(\lambda, \xi), \quad (3.48)$$

where

$$\begin{aligned} Z_1(\lambda, \xi) &= (\lambda^{-1}P_d + (\lambda P_r - Q(\xi))^{-1}P_r)(I + Y_1(\lambda, \xi))^{-1}Y_1(\lambda, \xi), \\ Y_1(\lambda, \xi) &= iP_d(v \cdot \xi)P_r(\lambda P_r - Q(\xi))^{-1}P_r + i\lambda^{-1}P_r(v \cdot \xi)(1 + \frac{1}{|\xi|^2})P_d. \end{aligned}$$

Substituting (3.48) into (3.47), we have the following decomposition of the semigroup $e^{t\hat{B}(\xi)}$ for $|\xi| \leq r_0$

$$e^{t\hat{B}(\xi)}f = e^{tQ(\xi)}P_rf - \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} e^{\lambda t} Z_2(\lambda, \xi) f d\lambda, \quad (3.49)$$

with $Z_2(\lambda, \xi) = Z_1(\lambda, \xi) - \lambda^{-1}P_d$. Denote

$$U_{\kappa, l} = \int_{-l}^l e^{(\kappa+iy)t} Z_2(\kappa+iy, \xi) f dy 1_{|\xi| \leq r_0}, \quad (3.50)$$

where the constant $l > 0$ is chosen large enough so that $l > y_1$ with y_1 defined in Lemma 3.8. Since the operator $Z_2(\lambda, \xi) = (\lambda P_r - Q(\xi))^{-1}P_r - (\lambda - \hat{B}(\xi))^{-1}$ is analytic on the domain $\text{Re } \lambda > -a_1$ with $a_1 > 0$ given by Lemma 3.8, we can shift the integration $U_{\kappa, l}$ from the line $\text{Re } \lambda = \kappa > 0$ to $\text{Re } \lambda = -\frac{a_1}{2}$ to deduce

$$U_{\kappa, l} = U_{-\frac{a_1}{2}, l} + H_l, \quad (3.51)$$

where

$$H_l = \left(\int_{-\frac{a_1}{2}+il}^{\kappa+il} - \int_{-\frac{a_1}{2}-il}^{\kappa-il} \right) e^{\lambda t} Z_2(\lambda, \xi) f d\lambda 1_{|\xi| \leq r_0}.$$

By Lemma 3.5, it can be seen that

$$\|H_l\|_{\xi} \rightarrow 0, \quad \text{as } l \rightarrow \infty. \quad (3.52)$$

By Cauchy Theorem, we have

$$\lim_{l \rightarrow \infty} \left| \int_{-\frac{a_1}{2}-il}^{-\frac{a_1}{2}+il} e^{\lambda t} \lambda^{-1} d\lambda \right| = 0,$$

which gives rise to

$$\lim_{l \rightarrow \infty} U_{-\frac{a_1}{2}, l}(t) = U_{-\frac{a_1}{2}, \infty}(t) =: \int_{-\frac{a_1}{2}-i\infty}^{-\frac{a_1}{2}+i\infty} e^{\lambda t} Z_1(\lambda, \xi) f d\lambda. \quad (3.53)$$

By Lemma 3.9, it holds that $\sup_{|\xi| \leq r_0, y \in \mathbb{R}} \|[I + Y_1(-\frac{a_1}{2} + iy, \xi)]^{-1}\|_{\xi} \leq C$. Thus, we have for any $f, g \in L_{\xi}^2(\mathbb{R}_v^3)$ that

$$\begin{aligned} |(U_{-\frac{a_1}{2}, \infty}(t)f, g)_{\xi}| &\leq C e^{-\frac{a_1 t}{2}} \int_{-\infty}^{+\infty} (\|[\lambda P_r - Q(\xi)]^{-1}P_r f\| + |\lambda|^{-1}\|P_d f\|_{\xi}) \\ &\quad \times (\|[\bar{\lambda} P_r - Q(-\xi)]^{-1}P_r g\| + |\bar{\lambda}|^{-1}\|P_d g\|_{\xi}) dy, \quad \lambda = -\frac{a_1}{2} + iy. \end{aligned}$$

This together with (3.45) yields $|(U_{-\frac{a_1}{2}, \infty}(t)f, g)_{\xi}| \leq C e^{-\frac{a_1 t}{2}} \|f\|_{\xi} \|g\|_{\xi}$, and

$$\|U_{-\frac{a_1}{2}, \infty}(t)\|_{\xi} \leq C e^{-\frac{a_1 t}{2}}. \quad (3.54)$$

Therefore, we conclude from (3.49)–(3.54) that

$$e^{t\hat{B}(\xi)}f = e^{tQ(\xi)}P_rf + U_{-\frac{a_1}{2}, \infty}(t), \quad |\xi| \leq r_0. \quad (3.55)$$

By (3.34), we have for $|\xi| > r_0$ that

$$(\lambda - \hat{B}(\xi))^{-1} = (\lambda - c(\xi))^{-1} + Z_3(\lambda, \xi), \quad (3.56)$$

where

$$\begin{aligned} Z_3(\lambda, \xi) &= (\lambda - c(\xi))^{-1} [I - Y_2(\lambda, \xi)]^{-1} Y_2(\lambda, \xi), \\ Y_2(\lambda, \xi) &= (K_1 - i(v \cdot \xi)|\xi|^{-2} P_d)(\lambda - c(\xi))^{-1}. \end{aligned}$$

Substituting (3.56) into (3.47) yields

$$e^{t\hat{B}(\xi)} f = e^{tc(\xi)} f + \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} e^{\lambda t} Z_3(\lambda, \xi) f d\lambda. \quad (3.57)$$

Denote

$$V_{\kappa, l} = \int_{-l}^l e^{(\kappa+iy)t} Z_3(\kappa+iy, \xi) dy 1_{|\xi|>r_0} \quad (3.58)$$

for sufficiently large constant $l > 0$ as in (3.50). Since the operator $Z_3(\lambda, \xi)$ is analytic on the domain $\operatorname{Re} \lambda \geq -a_1$, we can again shift the integration $V_{\kappa, l}$ from the line $\operatorname{Re} \lambda = \kappa > 0$ to $\operatorname{Re} \lambda = -\frac{a_1}{2}$, to deduce

$$V_{\kappa, l} = V_{-\frac{a_1}{2}, l} + I_l, \quad (3.59)$$

where

$$I_l = \left(\int_{-\frac{a_1}{2}+il}^{-\kappa+il} - \int_{-\frac{a_1}{2}-il}^{-\kappa-il} \right) e^{\lambda t} Z_3(\lambda, \xi) f d\lambda 1_{|\xi|>r_0}.$$

By Lemma 3.3, it can be shown that

$$\|I_l\| \rightarrow 0, \quad \text{as } l \rightarrow \infty. \quad (3.60)$$

Using the facts that (see Lemma 2.2.13 in [17] and Lemma 3.3 in [9])

$$\int_{-\infty}^{+\infty} \|(x+iy-c(\xi))^{-1} f\|^2 dy \leq \pi(x+\nu_0)^{-1} \|f\|^2, \quad \sup_{|\xi|>r_0, y \in \mathbb{R}} \|[I - Y_2(-\frac{a_1}{2}+iy, \xi)]^{-1}\| \leq C,$$

we have

$$\begin{aligned} |(V_{-\frac{a_1}{2}, \infty}(t), g)| &\leq C(\|K_1\| + r_0^{-1}) e^{-\frac{a_1}{2}t} \int_{-\infty}^{+\infty} \|(\lambda - c(\xi))^{-1} f\| \|(\bar{\lambda} - c(-\xi))^{-1} g\| dy \\ &\leq C e^{-\frac{a_1}{2}t} (\nu_0 - \frac{a_1}{2})^{-1} \|f\| \|g\|, \quad \lambda = -\frac{a_1}{2} + iy. \end{aligned} \quad (3.61)$$

By (3.61) and the fact $\|f\|^2 \leq \|f\|_{\xi}^2 \leq (1+r_0^{-2})\|f\|^2$ for $|\xi| > r_0$, we have

$$\|V_{-\frac{a_1}{2}, \infty}(t)\|_{\xi} \leq C e^{-\frac{a_1}{2}t} (\nu_0 - \frac{a_1}{2})^{-1}. \quad (3.62)$$

We conclude from (3.57) and (3.58)–(3.62) that

$$e^{t\hat{B}(\xi)} f = e^{tc(\xi)} f 1_{\{|\xi|>r_0\}} + V_{-\frac{a_1}{2}, \infty}(t), \quad |\xi| > r_0. \quad (3.63)$$

Finally, it follows from (3.55) and (3.63) that

$$e^{t\hat{B}(\xi)} f = (e^{tQ(\xi)} P_r f + U_{-\frac{a_1}{2}, \infty}(t)) 1_{\{|\xi| \leq r_0\}} + (e^{tc(\xi)} f + V_{-\frac{a_1}{2}, \infty}(t)) 1_{\{|\xi| > r_0\}}. \quad (3.64)$$

In particular, $e^{t\hat{B}(\xi)} f$ satisfies (3.46) in terms of (3.44), (3.54), (3.62) and the estimate $\|e^{tc(\xi)} 1_{\{|\xi|>r_0\}}\|_{\xi} \leq C e^{-\nu_0 t}$ because of (3.5) and (2.26). \square

Define a Sobolev space of function $f = f(x, v)$ by $H_P^N = \{f \in L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3) \mid \|f\|_{H_P^N} < \infty\}$ ($L_P^2 = H_P^0$) with the norm $\|\cdot\|_{H_P^N}$ defined by

$$\|f\|_{H_P^N} = \left(\int_{\mathbb{R}^3} (1 + |\xi|^2)^N \|\hat{f}\|_{\xi}^2 d\xi \right)^{1/2}$$

$$= \left(\int_{\mathbb{R}^3} (1 + |\xi|^2)^N \left(\int_{\mathbb{R}^3} |\hat{f}|^2 dv + \frac{1}{|\xi|^2} \left| \int_{\mathbb{R}^3} \hat{f} \sqrt{M} dv \right|^2 \right) d\xi \right)^{1/2},$$

where $\hat{f} = \hat{f}(\xi, v)$ is the Fourier transformation of $f(x, v)$. Note that $\|f\|_{H_P^N}^2 = \|f\|_{L_v^2(H_x^N)}^2 + \|\nabla_x \Delta_x^{-1}(f, \sqrt{M})\|_{H_x^N}^2$. For any $f_0 \in H_P^N$, we denote $e^{tB} f_0$ by

$$e^{tB} f_0 = (\mathcal{F}^{-1} e^{t\hat{B}(\xi)} \mathcal{F}) f_0. \quad (3.65)$$

By Lemma 3.1, it holds

$$\|e^{tB} f_0\|_{H_P^N} = \int_{\mathbb{R}^3} (1 + |\xi|^2)^N \|e^{t\hat{B}(\xi)} \hat{f}_0\|_{\xi}^2 d\xi \leq \int_{\mathbb{R}^3} (1 + |\xi|^2)^N \|\hat{f}_0\|_{\xi}^2 d\xi = \|f_0\|_{H_P^N}.$$

This means that the linear operator B generates a strongly continuous contraction semigroup e^{tB} in H_P^N , and therefore $f(x, v, t) = e^{tB} f_0(x, v)$ is a global solution to the IVP (2.28) for any $f_0 \in H_P^N$. What left is to establish the time-decay rates of the global solution.

Proof of Theorem 2.2. The property (2.33) of the spectrum to the operator $\hat{B}(\xi)$ follow from Lemma 3.8. And the proof of (2.34) and (2.35) can be found in [6, 15]. \square

Proof of Theorem 2.2. First, the proof of (2.36)–(2.39) can be found in [22]. Next, we prove (2.40). By (3.46) and using the fact that

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{(\xi^\alpha)^2}{|\xi|^2} \left| (\hat{f}_0, \sqrt{M}) \right|^2 d\xi &\leq \sup_{|\xi| \leq 1} \left| (\hat{f}_0, \sqrt{M}) \right|^2 \int_{|\xi| \leq 1} \frac{1}{|\xi|^2} d\xi + \int_{|\xi| > 1} (\xi^\alpha)^2 \left| (\hat{f}_0, \sqrt{M}) \right|^2 d\xi \\ &\leq C(\|(f_0, \sqrt{M})\|_{L_x^1}^2 + \|\partial_x^\alpha f_0\|_{L_{x,v}^2}^2), \end{aligned}$$

we have

$$\begin{aligned} \|\partial_x^\alpha e^{tB} f_0\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2 &= \int_{\mathbb{R}^3} (\xi^\alpha)^2 (\|e^{t\hat{B}(\xi)} \hat{f}_0\|_{L_v^2}^2 + \frac{1}{|\xi|^2} |(e^{t\hat{B}(\xi)} \hat{f}_0, \sqrt{M})|^2) d\xi \\ &\leq C \int_{\mathbb{R}^3} (\xi^\alpha)^2 e^{-a_1 t} (\|\hat{f}_0\|_{L_v^2}^2 + \frac{1}{|\xi|^2} |(\hat{f}_0, \sqrt{M})|^2) d\xi \\ &\leq C e^{-a_1 t} (\|f_0\|_{L_{2,1}^2}^2 + \|\partial_x^\alpha f_0\|_{L_{x,v}^2}^2). \end{aligned}$$

This proves the theorem. \square

3.3 Analysis of spectrum and semigroup for linear mVPB system

We establish in this subsection the analysis of spectrum and resolvent and investigate the long time properties of the semigroup related the linear operator B_m to the mVPB (2.61) by applying the similar arguments as the above.

Introduce a weighted Hilbert space $L_m^2(\mathbb{R}^3)$ as

$$L_m^2(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3) \mid \|f\|_m = \sqrt{\langle f, f \rangle_\xi} < \infty\}$$

equipped with the inner product

$$\langle f, g \rangle_\xi = (f, g) + \frac{1}{1 + |\xi|^2} (P_d f, P_d g).$$

We have for any $f, g \in L_\xi^2(\mathbb{R}_v^3) \cap D(\hat{B}_m(\xi))$,

$$\langle \hat{B}_m(\xi) f, g \rangle_\xi = (\hat{B}_m(\xi) f, g) + \frac{1}{1 + |\xi|^2} P_d g = (f, (L + i(v \cdot \xi) + \frac{i(v \cdot \xi)}{1 + |\xi|^2} P_d) g) = \langle f, \hat{B}_m(-\xi) g \rangle_\xi.$$

Since

$$\|f\| \leq \|f\|_m \leq 2\|f\|, \quad \forall \xi \in \mathbb{R}^3,$$

we can regard $\hat{B}(\xi)$ as an operator from $L^2(\mathbb{R}^3)$ to itself or from $L_\xi^2(\mathbb{R}^3)$ to itself. Then, similar to Lemma 3.1 and 3.2, we have the following two lemmas.

Lemma 3.12. *The operator $\hat{B}_m(\xi)$ generates a strongly continuous contraction semigroup on $L_m^2(\mathbb{R}_v^3)$ satisfying*

$$\|e^{t\hat{B}_m(\xi)} f\|_m \leq \|f\|_m \quad \text{for any } t > 0, f \in L_m^2(\mathbb{R}_v^3).$$

Lemma 3.13. *For each $\xi \in \mathbb{R}^3$, the spectrum $\lambda \in \sigma(\hat{B}_m(\xi))$ on the domain $\operatorname{Re}\lambda \geq -\nu_0 + \delta$ for any constant $\delta > 0$ consists of isolated eigenvalues with $\operatorname{Re}\lambda(\xi) < 0$ for $\xi \neq 0$ and $\lambda(\xi) = 0$ only if $\xi = 0$.*

Next, we deal with the spectrum and resolvent sets of $\hat{B}_m(\xi)$. To this end, we decompose $\hat{B}_m(\xi)$ into

$$\begin{aligned} \lambda - \hat{B}_m(\xi) &= \lambda - c(\xi) - K + \frac{i(v \cdot \xi)}{1 + |\xi|^2} P_d \\ &= (I - K(\lambda - c(\xi))^{-1} + \frac{i(v \cdot \xi)}{1 + |\xi|^2} P_d(\lambda - c(\xi))^{-1})(\lambda - c(\xi)), \end{aligned} \quad (3.66)$$

where $c(\xi)$ is defined by (3.5). Then, we have the estimates on the right hand terms of (3.66) as follows.

Lemma 3.14. *There exists a constant $C > 0$ so that it holds:*

1. *For any $\delta > 0$, we have*

$$\sup_{x \geq -\nu_0 + \delta, y \in \mathbb{R}} \|K(x + iy - c(\xi))^{-1}\| \leq C\delta^{-15/13}(1 + |\xi|)^{-2/13}, \quad (3.67)$$

2. *For any $\delta > 0$, $r_0 > 0$, there is a constant $y_0 = (2r_0)^{5/3}\delta^{-2/3} > 0$ such that if $|y| \geq y_0$, we have*

$$\sup_{x \geq -\nu_0 + \delta, |\xi| \leq r_0} \|K(x + iy - c(\xi))^{-1}\| \leq C\delta^{-7/5}(1 + |y|)^{-2/5}, \quad (3.68)$$

3. *For any $\delta > 0$, we have*

$$\sup_{x \geq -\nu_0 + \delta, y \in \mathbb{R}} \|(v \cdot \xi)(1 + |\xi|^2)^{-1} P_d(x + iy - c(\xi))^{-1}\| \leq C\delta^{-1}|\xi|(1 + |\xi|^2)^{-1}, \quad (3.69)$$

$$\sup_{x \geq -\nu_0 + \delta, \xi \in \mathbb{R}^3} \|(v \cdot \xi)(1 + |\xi|^2)^{-1} P_d(x + iy - c(\xi))^{-1}\| \leq C(\delta^{-1} + 1)|y|^{-1}. \quad (3.70)$$

Proof. The proof of (3.67) and (3.68) can be done as that of Lemma 2.2.6 in [17], we omit the details. The (3.69) can be obtained by virtue of $\|(v \cdot \xi)(1 + |\xi|^2)^{-1} P_d\| \leq C|\xi|(1 + |\xi|^2)^{-1}$ and $\|(x + iy - c(\xi))^{-1}\| \leq \delta^{-1}$ for $x \geq -\nu_0 + \delta$. The (3.70) follows from the facts $\frac{(v \cdot \xi)}{1 + |\xi|^2} P_d(\lambda - c(\xi))^{-1} = \frac{1}{\lambda} \frac{(v \cdot \xi)}{1 + |\xi|^2} P_d + \frac{1}{\lambda} \frac{(v \cdot \xi)}{1 + |\xi|^2} P_d c(\xi)(\lambda - c(\xi))^{-1}$ and $\|\frac{(v \cdot \xi)}{1 + |\xi|^2} P_d c(\xi)\| \leq C$. \square

Lemma 3.15. *It holds.*

1. *For any $\delta > 0$ and all $\xi \in \mathbb{R}^3$, there exists $y_1(\delta) > 0$ such that*

$$\rho(\hat{B}_m(\xi)) \supset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq -\nu_0 + \delta, |\operatorname{Im}\lambda| \geq y_1\} \cup \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}. \quad (3.71)$$

2. *For any $r_0 > 0$, there exists $\alpha = \alpha(r_0) > 0$ such that it holds for $|\xi| \geq r_0$ that*

$$\sigma(\hat{B}_m(\xi)) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leq -\alpha\}. \quad (3.72)$$

3. *For any $\delta > 0$, there exists $r_1(\delta) > 0$ such that if $0 < |\xi| \leq r_1$, then*

$$\sigma(\hat{B}_m(\xi)) \cap \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq -\mu/2\} \subset \{\lambda \in \mathbb{C} \mid |\lambda| < \delta\}. \quad (3.73)$$

Proof. By (3.66) and Lemma 3.14, we can prove (3.71). The properties (3.72) and (3.73) can be obtained by the similar argument as Proposition 2.3 in [6]. \square

By applying the basic ideas similar to those used in Section 2.2 in [9], we can make a detailed analysis of the spectral of the operator $\hat{B}_m(\xi)$ at lower frequency below, the details of the proof are omitted.

Lemma 3.16. *There exists a constant $r_0 > 0$ so that the spectrum $\lambda \in \sigma(B_m(\xi)) \subset \mathbb{C}$ for $\xi = s\omega$ with $|s| \leq r_0$ and $\omega \in \mathbb{S}^2$ consists of five points $\{\lambda_j(s), j = -1, 0, 1, 2, 3\}$ on the domain $\text{Re}\lambda > -\mu/2$. The spectrum $\lambda_j(s)$ and the corresponding eigenfunction $\psi_j(s, \omega)$ are C^∞ functions of s for $|s| \leq r_0$. In particular, the eigenvalues admit the following asymptotical expansion for $|s| \leq r_0$*

$$\begin{cases} \lambda_{\pm 1}(s) = \pm i2\sqrt{\frac{2}{3}}s - a_{\pm 1}s^2 + o(s^2), & \overline{\lambda_1(s)} = \lambda_{-1}(s), \\ \lambda_0(s) = -a_0s^2 + o(s^2), \\ \lambda_2(s) = \lambda_3(s) = -a_2s^2 + o(s^2), \end{cases} \quad (3.74)$$

with $a_j > 0$, $-1 \leq j \leq 2$, are defined by

$$\begin{cases} a_{\pm 1} = -\frac{1}{8}(L^{-1}P_1(v_1\chi_4), v_1\chi_4) - \frac{1}{2}(L^{-1}P_1(v_1\chi_1), v_1\chi_1), \\ a_0 = -\frac{3}{4}(L^{-1}P_1(v_1\chi_4), v_1\chi_4), \quad a_2 = -(L^{-1}P_1(v_1\chi_2), v_1\chi_2). \end{cases} \quad (3.75)$$

The eigenfunctions are orthogonal each other and satisfy

$$\begin{cases} \langle \psi_j(s, \omega), \overline{\psi_k(s, \omega)} \rangle_\xi = \delta_{jk}, \quad j, k = -1, 0, 1, 2, 3, \\ \psi_j(s, \omega) = \psi_{j,0}(\omega) + \psi_{j,1}(\omega)s + O(s^2), \quad |s| \leq r_0, \end{cases} \quad (3.76)$$

where the coefficients $\psi_{j,n}$ are given as

$$\begin{cases} \psi_{0,0} = \frac{\sqrt{2}}{4}\sqrt{M} - \frac{\sqrt{3}}{2}\chi_4, & P_1(\psi_{0,1}) = iL^{-1}P_1(v \cdot \omega)\psi_{0,0}; \\ \psi_{\pm 1,0} = \frac{\sqrt{3}}{4}\sqrt{M} \mp \frac{\sqrt{2}}{2}(v \cdot \omega)\sqrt{M} + \frac{\sqrt{2}}{4}\chi_4, & P_1(\psi_{\pm 1,1}) = iL^{-1}P_1(v \cdot \omega)\psi_{\pm 1,0}, \\ \psi_{j,0} = (v \cdot W^j)\sqrt{M}, & P_1(\psi_{j,1}) = iL^{-1}P_1(v \cdot \omega)(v \cdot W^j)\sqrt{M}, \quad j = 2, 3, \end{cases} \quad (3.77)$$

and W^j ($j = 2, 3$) are orthogonal vectors satisfying $W^j \cdot \omega = 0$.

Proof of Theorem 2.5. The combination of Lemmas 3.15–3.16 leads to Theorem 2.5. \square

Remark 3.17. In general, the electric potential equation in (2.58) takes as $a\Delta_x\Phi = \int_{\mathbb{R}^3} f\sqrt{M}dv - e^{-b\Phi}$ with two constants $a, b > 0$, and the asymptotical expansion of the eigenvalues $\lambda_j(s)$ becomes

$$\begin{cases} \lambda_{\pm 1}(s) = \pm i\sqrt{\frac{1}{b} + \frac{5}{3}}s - a_{\pm 1}s^2 + o(s^2), & \overline{\lambda_1(s)} = \lambda_{-1}(s), \\ \lambda_0(s) = -a_0s^2 + o(s^2), \\ \lambda_2(s) = \lambda_3(s) = -a_2s^2 + o(s^2), \\ a_{\pm 1} = -\frac{b}{5b+3}(L^{-1}P_1(v_1\chi_4), v_1\chi_4) - \frac{1}{2}(L^{-1}P_1(v_1\chi_1), v_1\chi_1), \\ a_0 = -\frac{3(b+1)}{5b+3}(L^{-1}P_1(v_1\chi_4), v_1\chi_4), \quad a_2 = -(L^{-1}P_1(v_1\chi_2), v_1\chi_2). \end{cases}$$

With the help of Lemmas 3.13–3.16, we can make a detailed analysis on the semigroup $S(t, \xi) = e^{t\hat{B}_m(\xi)}$ with respect to the lower frequency and higher frequency in terms of an argument similar to that of Theorem 3.11, we omit the details.

Theorem 3.18. *The semigroup $S(t, \xi) = e^{t\hat{B}_m(\xi)}$ with $\xi = s\omega \in \mathbb{R}^3$, $s = |\xi|$ has the following decomposition*

$$S(t, \xi)f = S_1(t, \xi)f + S_2(t, \xi)f, \quad f \in L_m^2(\mathbb{R}_v^3), \quad t \geq 0, \quad (3.78)$$

where

$$S_1(t, \xi)f = \sum_{j=-1}^3 e^{t\lambda_j(s)} \langle f, \overline{\psi_j(s, \omega)} \rangle_{\xi} \psi_j(s, \omega) 1_{\{|\xi| \leq r_0\}}, \quad (3.79)$$

and $S_2(t, \xi)f =: S(t, \xi)f - S_1(t, \xi)f$ satisfies

$$\|S_2(t, \xi)f\|_m \leq Ce^{-\sigma_0 t} \|f\|_m, \quad t \geq 0, \quad (3.80)$$

with $\sigma_0 > 0$ a constant independent of ξ .

3.4 Optimal time-decay rates for linear mVPB

With the help of the spectral analysis and semigroup estimates in section 3.3, we can prove Theorems 2.6 on the optimal time decay rate of global solution.

Proof of Theorem 2.6. Write $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \xi_3^{\alpha_3}$ with $\xi = (\xi_1, \xi_2, \xi_3)$. By Theorem 3.18 and the Planchel's equality, we have

$$\begin{aligned} \|(\partial_x^\alpha e^{tB_m} f_0, \chi_j)\|_{L_x^2} &\leq \|\xi^\alpha (S_1(t, \xi) \hat{f}_0, \chi_j)\|_{L_\xi^2} + \|\xi^\alpha (S_2(t, \xi) \hat{f}_0, \chi_j)\|_{L_\xi^2} \\ &\leq \|\xi^\alpha (S_1(t, \xi) \hat{f}_0, \chi_j)\|_{L_\xi^2} + \|\xi^\alpha S_2(t, \xi) \hat{f}_0\|_{L_{\xi, v}^2}, \end{aligned} \quad (3.81)$$

$$\|P_1(\partial_x^\alpha e^{tB_m} f_0)\|_{L_{x, v}^2} \leq \|\xi^\alpha P_1(S_1(t, \xi) \hat{f}_0)\|_{L_{\xi, v}^2} + \|\xi^\alpha S_2(t, \xi) \hat{f}_0\|_{L_{\xi, v}^2}. \quad (3.82)$$

By (3.80) we obtain

$$\int_{\mathbb{R}^3} (\xi^\alpha)^2 \|S_2(t, \xi) \hat{f}_0\|_m^2 d\xi \leq C \int_{\mathbb{R}^3} e^{-2\sigma_0 t} (\xi^\alpha)^2 \|\hat{f}_0\|_m^2 d\xi \leq Ce^{-2\sigma_0 t} \|\partial_x^\alpha f\|_{L_{x, v}^2}^2. \quad (3.83)$$

By (3.76) and (3.79), we have for $|\xi| \leq r_0$ that

$$S_1(t, \xi) \hat{f}_0 = \sum_{j=-1}^3 e^{t\lambda_j(|\xi|)} [\langle \hat{f}_0, \overline{\psi_{j,0}} \rangle_{\xi=0} \psi_{j,0} + |\xi| T_j(\xi) \hat{f}_0],$$

where $\lambda_j(|\xi|)$ is given by (3.74), $\langle f, g \rangle_{\xi=0} = (f, g) + (P_d f, P_d g)$ and $T_j(\xi)$ is the linear operator with the norm $\|T_j(\xi)\|$ uniformly bounded for $|\xi| \leq r_0$ and $-1 \leq j \leq 3$. Therefore

$$\begin{aligned} (S_1(t, \xi) \hat{f}_0, \sqrt{M}) &= \frac{\sqrt{3}}{4} \sum_{j=\pm 1} e^{\lambda_j(|\xi|)t} \left[\frac{\sqrt{3}}{2} \hat{n}_0 - j \frac{\sqrt{2}}{2} (\hat{m}_0 \cdot \omega) + \frac{\sqrt{2}}{4} \hat{q}_0 \right] \\ &\quad + \frac{\sqrt{2}}{4} e^{\lambda_0(|\xi|)t} \left(\frac{\sqrt{2}}{2} \hat{n}_0 - \frac{\sqrt{3}}{2} \hat{q}_0 \right) + |\xi| \sum_{j=-1}^3 e^{\lambda_j(|\xi|)t} (T_j(\xi) \hat{f}_0, \sqrt{M}), \end{aligned} \quad (3.84)$$

$$\begin{aligned} (S_1(t, \xi) \hat{f}_0, v\sqrt{M}) &= -\frac{\sqrt{2}}{2} \sum_{j=\pm 1} e^{\lambda_j(|\xi|)t} j \left[\frac{\sqrt{3}}{2} \hat{n}_0 - j \frac{\sqrt{2}}{2} (\hat{m}_0 \cdot \omega) + \frac{\sqrt{2}}{4} \hat{q}_0 \right] \omega \\ &\quad + \sum_{j=2,3} e^{\lambda_j(|\xi|)t} (\hat{m}_0 \cdot W^j) W^j + |\xi| \sum_{j=-1}^3 e^{\lambda_j(|\xi|)t} (T_j(\xi) \hat{f}_0, v\sqrt{M}), \end{aligned} \quad (3.85)$$

$$\begin{aligned} (S_1(t, \xi) \hat{f}_0, \chi_4) &= \frac{\sqrt{2}}{4} \sum_{j=\pm 1} e^{\lambda_j(|\xi|)t} \left[\frac{\sqrt{3}}{2} \hat{n}_0 - j \frac{\sqrt{2}}{2} (\hat{m}_0 \cdot \omega) + \frac{\sqrt{2}}{4} \hat{q}_0 \right] \\ &\quad - \frac{\sqrt{3}}{2} e^{\lambda_0(|\xi|)t} \left(\frac{\sqrt{2}}{2} \hat{n}_0 - \frac{\sqrt{3}}{2} \hat{q}_0 \right) + |\xi| \sum_{j=-1}^3 e^{\lambda_j(|\xi|)t} (T_j(\xi) \hat{f}_0, \chi_4), \end{aligned} \quad (3.86)$$

where we recall that $\lambda_j(|\xi|)$ is given by (3.74), $(\hat{n}_0, \hat{m}_0, \hat{q}_0)$ is the Fourier transform of the macroscopic density, momentum and energy (n_0, m_0, q_0) of the initial data f_0 defined by $(n_0, m_0, q_0) = ((f_0, \chi_0), (f_0, v\sqrt{M}), (f_0, \chi_4))$ and W^j is given by (3.77), and

$$P_1(S_1(t, \xi)\hat{f}_0) = |\xi| \sum_{j=-1}^3 e^{\lambda_j(|\xi|)t} P_1(T_j(\xi)\hat{f}_0). \quad (3.87)$$

Noting by (3.74) that

$$\operatorname{Re}\lambda_j(|\xi|) = a_j|\xi|^2(1 + O(|\xi|)) \leq -\beta|\xi|^2, \quad |\xi| \leq r_0, \quad (3.88)$$

where and below $\beta > 0$ denotes a generic constant, we obtain by (3.84)–(3.87) that

$$|(S_1(t, \xi)\hat{f}_0, \chi_j)|^2 \leq C e^{-2\beta|\xi|^2 t} (|\hat{n}_0, \hat{m}_0, \hat{q}_0|^2 + |\xi|^2 \|\hat{f}_0\|_{L_v^2}^2), \quad j = 0, 1, 2, 3, 4, \quad (3.89)$$

$$\|P_1(S_1(t, \xi)\hat{f}_0)\|_{L_v^2}^2 \leq C|\xi|^2 e^{-2\beta|\xi|^2 t} \|\hat{f}_0\|_{L_v^2}^2. \quad (3.90)$$

Thus by (3.89), Hölder and Hausdorff-Young inequalities, we have

$$\|\xi^\alpha(S_1(t, \xi)\hat{f}_0, \chi_j)\|_{L_\xi^2}^2 \leq C(1+t)^{-3(\frac{1}{q}-\frac{1}{2})-k} \|\partial_x^{\alpha'} f_0\|_{L^{2,q}}^2,$$

with $k = |\alpha - \alpha'|$. This proves (2.65). Similarly, by (3.89) and (3.90) we can prove (2.66) and (2.67).

Now we turn to show the lower bound of time-decay rates for the global solution under the assumptions of Theorem 2.6. Note that

$$\begin{aligned} \|\nabla_x^k(e^{tB_m} f_0, \chi_j)\|_{L_x^2} &\geq \| |\xi|^k (S_1(t, \xi)\hat{f}_0, \chi_j) \|_{L_\xi^2} - \| |\xi|^k S_2(t, \xi)\hat{f}_0 \|_{L_{\xi,v}^2} \\ &\geq \| |\xi|^k (S_1(t, \xi)\hat{f}_0, \chi_j) \|_{L_\xi^2} - C e^{-\sigma_0 t} \|\nabla_x^k f_0\|_{L_{x,v}^2}, \quad j = 0, 1, 2, 3, 4, \end{aligned} \quad (3.91)$$

$$\begin{aligned} \|\nabla_x^k P_1(e^{tB_m} f_0)\|_{L_{x,v}^2}^2 &\geq \| |\xi|^k P_1(S_1(t, \xi)\hat{f}_0) \|_{L_{\xi,v}^2}^2 - \| |\xi|^k S_2(t, \xi)\hat{f}_0 \|_{L_{\xi,v}^2}^2 \\ &\geq \| |\xi|^k P_1(S_1(t, \xi)\hat{f}_0) \|_{L_{\xi,v}^2}^2 - C e^{-\sigma_0 t} \|\nabla_x^k f_0\|_{L_{x,v}^2}^2, \end{aligned} \quad (3.92)$$

where we have used (3.83) for $|\alpha| = k$.

First, we prove (2.68)–(2.70) as follows. By (3.84) and (3.88), we have

$$\begin{aligned} |(S_1(t, \xi)\hat{f}_0, \chi_0)|^2 &\geq \frac{1}{16} |\hat{q}_0|^2 \left| e^{\operatorname{Re}\lambda_1(|\xi|)t} \cos(\operatorname{Im}\lambda_1(|\xi|)t) - e^{\lambda_0(|\xi|)t} \right|^2 - e^{-2\beta|\xi|^2 t} (2|\hat{n}_0|^2 + C|\xi|^2 \|\hat{f}_0\|_{L_v^2}^2) \\ &\geq \frac{1}{20} |\hat{q}_0|^2 \left| e^{\operatorname{Re}\lambda_1(|\xi|)t} \cos\left(2\sqrt{\frac{2}{3}}|\xi|t\right) - e^{\lambda_0(|\xi|)t} \right|^2 \\ &\quad - e^{-2\beta|\xi|^2 t} (C(|\xi|^3 t)^2 |\hat{q}_0|^2 + 2|\hat{n}_0|^2 + C|\xi|^2 \|\hat{f}_0\|_{L_v^2}^2), \end{aligned} \quad (3.93)$$

due to the fact $\cos(\operatorname{Im}\lambda_1(|\xi|)t) \sim \cos\left(2\sqrt{\frac{2}{3}}|\xi|t\right) + O(|\xi|^3 t)$, which implies

$$\begin{aligned} \| |\xi|^k (S_1(t, \xi)\hat{f}_0, \chi_0) \|_{L_\xi^2}^2 &\geq \frac{1}{20} \inf_{|\xi| \leq r_0} |\hat{q}_0|^2 \int_{|\xi| \leq r_0} |\xi|^{2k} \left| e^{\operatorname{Re}\lambda_1(|\xi|)t} \cos\left(2\sqrt{\frac{2}{3}}|\xi|t\right) - e^{\lambda_0(|\xi|)t} \right|^2 d\xi \\ &\quad - C_3 \sup_{|\xi| \leq r_0} |\hat{n}_0|^2 (1+t)^{-3/2-k} - C(\|q_0\|_{L_x^1}^2 + \|f_0\|_{L^{2,1}}^2)(1+t)^{-5/2-k} \\ &=: \frac{1}{20} \inf_{|\xi| \leq r_0} |\hat{q}_0|^2 I_1 - C_3 \sup_{|\xi| \leq r_0} |\hat{n}_0|^2 (1+t)^{-3/2-k} - C(1+t)^{-5/2-k}. \end{aligned} \quad (3.94)$$

For the term I_1 , it holds for $t \geq t_0 =: \frac{1}{r_0^2}$ that

$$I_1 \geq 4\pi \sum_{k=0}^{\lceil \frac{1}{4}\sqrt{\frac{2}{3}}r_0 t \rceil - 1} \int_{(2k\pi + \frac{1}{2}\pi)/(2\sqrt{\frac{2}{3}}t)}^{(2k\pi + \frac{3}{2}\pi)/(2\sqrt{\frac{2}{3}}t)} r^{2+2k} \left| e^{\operatorname{Re}\lambda_1(r)t} \cos\left(2\sqrt{\frac{2}{3}}rt\right) - e^{\lambda_0(r)t} \right|^2 dr$$

$$\geq 4\pi \sum_{k=0}^{[\frac{1}{\pi}\sqrt{\frac{2}{3}}r_0t]-1} \int_{(k\pi+\frac{1}{4}\pi)/(\sqrt{\frac{2}{3}}t)}^{(k\pi+\frac{3}{4}\pi)/(\sqrt{\frac{2}{3}}t)} r^{2+2k} e^{2\lambda_0(r)t} dr \geq \frac{1}{2}\pi e^{-2\eta}(1+t)^{-3/2-k}, \quad (3.95)$$

where we have used by (3.74)

$$\operatorname{Re}\lambda_j(|\xi|) = a_j|\xi|^2(1 + O(|\xi|)) \geq -\eta|\xi|^2, \quad |\xi| \leq r_0. \quad (3.96)$$

It follows from (3.94) and (3.95) that

$$\begin{aligned} \| |\xi|^k (S_1(t, \xi) \hat{f}_0, \chi_0) \|_{L_\xi^2}^2 &\geq C_4 \inf_{|\xi| \leq r_0} |q_0|^2 (1+t)^{-3/2-k} - C_3 \sup_{|\xi| \leq r_0} |\hat{n}_0|^2 (1+t)^{-3/2-k} \\ &\quad - C(1+t)^{-5/2-k}, \end{aligned} \quad (3.97)$$

which and (3.91) with $j = 0$ lead to (2.68) and (2.69) for $d_1 > \frac{C_3}{C_4} > 0$ and $t > 0$ large enough.

By (3.85), we obtain

$$|(S_1(t, \xi) \hat{f}_0, v\sqrt{M})|^2 \geq \frac{1}{8} e^{2\operatorname{Re}\lambda_1(|\xi|)t} \sin^2(\operatorname{Im}\lambda_1(|\xi|)t) \left(|\hat{q}_0| - \sqrt{6}|\hat{n}_0| \right)^2 - C|\xi|^2 e^{-2\beta|\xi|^2 t} \|\hat{f}_0\|_{L_v^2}^2. \quad (3.98)$$

In terms of (3.96) and the fact

$$\sin^2(\operatorname{Im}\lambda_1(|\xi|)t) \geq \frac{1}{2} \sin^2 \left(2\sqrt{\frac{2}{3}}|\xi|t \right) - O(|\xi|^3 t^2),$$

we obtain by (3.98) that

$$\begin{aligned} \| |\xi|^k (S_1(t, \xi) \hat{f}_0, v\sqrt{M}) \|_{L_\xi^2}^2 &\geq \frac{1}{16} (d_1 - \sqrt{6})^2 d_0^2 \int_{|\xi| \leq r_0} |\xi|^{2k} e^{-2\eta|\xi|^2 t} \sin^2 \left(2\sqrt{\frac{2}{3}}|\xi|t \right) d\xi \\ &\quad - C(\|n_0\|_{L_x^1}^2 + \|q_0\|_{L_x^1}^2 + \|f_0\|_{L^{2,1}}^2)(1+t)^{-5/2-k} \\ &=: \frac{1}{16} (d_1 - \sqrt{6})^2 d_0^2 I_2 - C(1+t)^{-5/2-k}. \end{aligned} \quad (3.99)$$

Since it holds for $t \geq t_0 =: \frac{L^2}{r_0^2}$ with the constant $L > 4\pi$ that

$$\begin{aligned} I_2 &= t^{-3/2-k} \int_{|\zeta| \leq r_0 \sqrt{t}} |\zeta|^{2k} e^{-2\eta|\zeta|^2} \sin^2 \left(2\sqrt{\frac{2}{3}}|\zeta|\sqrt{t} \right) d\zeta \\ &\geq \pi(1+t)^{-3/2-k} L^{2+2k} e^{-2\eta L^2} \int_{L/2}^L \sin^2 \left(2\sqrt{\frac{2}{3}}r\sqrt{t} \right) dr \\ &\geq \pi(1+t)^{-3/2-k} L^{2+2k} e^{-2\eta L^2} \int_0^\pi \sin^2 y dy > C_3(1+t)^{-3/2-k}, \end{aligned} \quad (3.100)$$

we obtain (2.68) for $t > 0$ large enough, with the help of (3.99) and (3.91) for $j = 1, 2, 3$.

By (3.86), we obtain

$$\begin{aligned} |(S_1(t, \xi) \hat{f}_0, \chi_4)|^2 &\geq \frac{1}{32} |\hat{q}_0|^2 e^{2\operatorname{Re}\lambda_1(|\xi|)t} \cos^2(\operatorname{Im}\lambda_1(|\xi|)t) \left| e^{\operatorname{Re}\lambda_1(|\xi|)t} \cos(\operatorname{Im}\lambda_1(|\xi|)t) + 3e^{\lambda_0(|\xi|)t} \right|^2 \\ &\quad - 2e^{-2\beta|\xi|^2 t} |\hat{n}_0|^2 - C|\xi|^2 e^{-2\beta|\xi|^2 t} \|\hat{f}_0\|_{L_v^2}^2, \end{aligned}$$

with which, we can obtain by the similar argument to estimate the terms in (3.95) that

$$\begin{aligned} \| |\xi|^k (S_1(t, \xi) \hat{f}_0, \chi_4) \|_{L_\xi^2}^2 &\geq C_4 \inf_{|\xi| \leq r_0} |q_0|^2 (1+t)^{-3/2-k} - C_3 \sup_{|\xi| \leq r_0} |\hat{n}_0|^2 (1+t)^{-3/2-k} \\ &\quad - C(1+t)^{-5/2-k}, \end{aligned}$$

which together with (3.91) for $j = 4$ lead to (2.68) for $t > 0$ large enough.

Next, we prove (2.70). By (3.87) and (3.77), we have

$$\begin{aligned} P_1(S_1(t, \xi)\hat{f}_0) &= i|\xi| \sum_{j=\pm 1} e^{\lambda_j(|\xi|)t} \left(\frac{\sqrt{3}}{2}\hat{n}_0 + \frac{\sqrt{2}}{4}\hat{q}_0 \right) \left[\frac{\sqrt{2}}{4}L^{-1}P_1(v \cdot \omega)\chi_4 - j\frac{\sqrt{2}}{2}L^{-1}P_1(v \cdot \omega)^2\sqrt{M} \right] \\ &\quad + i|\xi| \sum_{j=2,3} e^{\lambda_j(|\xi|)t} (\hat{m}_0 \cdot W^j) L^{-1}P_1(v \cdot \omega)(v \cdot W^j)\sqrt{M} \\ &\quad - i\frac{\sqrt{3}}{2}|\xi| e^{\lambda_0(|\xi|)t} \left(\frac{\sqrt{2}}{2}\hat{n}_0 - \frac{\sqrt{3}}{2}\hat{q}_0 \right) L^{-1}P_1(v \cdot \omega)\chi_4 + |\xi|^2 T_4(t, \xi)\hat{f}_0, \end{aligned}$$

where $T_4(t, \xi)\hat{f}_0$ is the remainder term satisfying $\|T_4(t, \xi)\hat{f}_0\|_{L_v^2}^2 \leq C e^{-2\beta|\xi|^2 t} \|\hat{f}_0\|_{L_v^2}^2$. Noting that the terms $L^{-1}P_1(v \cdot \omega)\chi_4$, $L^{-1}P_1(v \cdot \omega)^2\sqrt{M}$, and $L^{-1}P_1(v \cdot \omega)(v \cdot W^j)\sqrt{M}$ are orthogonal to each other, we have

$$\begin{aligned} \|P_1(S_1(t, \xi)\hat{f}_0)\|_{L_v^2}^2 &\geq \frac{1}{8} \|L^{-1}P_1(v_1\chi_1)\|_{L_v^2}^2 |\xi|^2 e^{2\operatorname{Re}\lambda_1(|\xi|)t} \sin^2(\operatorname{Im}\lambda_1(|\xi|)t) \left(|\hat{q}_0| - \sqrt{6}|\hat{n}_0| \right)^2 \\ &\quad - C|\xi|^4 e^{-2\beta|\xi|^2 t} \|\hat{f}_0\|_{L_v^2}^2, \end{aligned}$$

and then

$$\| |\xi|^k P_1(S_1(t, \xi)\hat{f}_0) \|_{L_{\xi,v}^2}^2 \geq C_4(1+t)^{-5/2-k} - C(1+t)^{-7/2-k}.$$

This together with (3.92) leads to (2.69) for $t > 0$ large enough. This completes the proof of the theorem. \square

4 The nonlinear problem for bVPB system

4.1 Energy estimates

Let N be a positive integer, and

$$\begin{aligned} E_k(f_1, f_2) &= \sum_{|\alpha|+|\beta|\leq N} \|w^k \partial_x^\alpha \partial_v^\beta(f_1, f_2)\|_{L_{x,v}^2}^2 + \sum_{|\alpha|\leq N} \|w^k \partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2, \\ H_k(f_1, f_2) &= \sum_{|\alpha|+|\beta|\leq N} \|w^k \partial_x^\alpha \partial_v^\beta(P_1 f_1, P_r f_2)\|_{L_{x,v}^2}^2 \\ &\quad + \sum_{|\alpha|\leq N-1} (\|\partial_x^\alpha \nabla_x(P_0 f_1, P_d f_2)\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha P_d f_2\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2), \\ D_k(f_1, f_2) &= \sum_{|\alpha|+|\beta|\leq N} \|w^{\frac{1}{2}+k} \partial_x^\alpha \partial_v^\beta(P_1 f_1, P_r f_2)\|_{L_{x,v}^2}^2 \\ &\quad + \sum_{|\alpha|\leq N-1} (\|\partial_x^\alpha \nabla_x(P_0 f_1, P_d f_2)\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha P_d f_2\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2). \end{aligned}$$

for $k \geq 0$. For brevity, we write $E(f_1, f_2) = E_0(f_1, f_2)$, $H(f_1, f_2) = H_0(f_1, f_2)$ and $D(f_1, f_2) = D_0(f_1, f_2)$ for $k = 0$.

Firstly, by taking inner product between χ_j ($j = 0, 1, 2, 3, 4$) and (2.14), we obtain the system of compressible Euler-Poisson type (EP) as

$$\partial_t n_1 + \operatorname{div}_x m_1 = 0, \tag{4.1}$$

$$\partial_t m_1 + \nabla_x n_1 + \sqrt{\frac{2}{3}} \nabla_x q_1 = n_2 \nabla_x \Phi - \int_{\mathbb{R}^3} v \cdot \nabla_x (P_1 f_1) v \sqrt{M} dv, \tag{4.2}$$

$$\partial_t q_1 + \sqrt{\frac{2}{3}} \operatorname{div}_x m_1 = \sqrt{\frac{2}{3}} \nabla_x \Phi \cdot m_2 - \int_{\mathbb{R}^3} v \cdot \nabla_x (P_1 f_1) \chi_4 dv, \tag{4.3}$$

where

$$(n_1, m_1, q_1) = ((f_1, \sqrt{M}), (f_1, v\sqrt{M}), (f_1, \chi_4)), \quad (n_2, m_2) = ((f_2, \sqrt{M}), (f_2, v\sqrt{M})).$$

Taking the microscopic projection P_1 to (2.14), we have

$$\partial_t(P_1 f_1) + P_1(v \cdot \nabla_x P_1 f_1) - L(P_1 f_1) = -P_1(v \cdot \nabla_x P_0 f_1) + P_1 G_1, \quad (4.4)$$

where the nonlinear term G_1 is denoted by

$$G_1 = \frac{1}{2}(v \cdot \nabla_x \Phi) f_2 - \nabla_x \Phi \cdot \nabla_v f_2 + \Gamma(f_1, f_1). \quad (4.5)$$

By (4.4), we can express the microscopic part $P_1 f_1$ as

$$P_1 f_1 = L^{-1}[\partial_t(P_1 f_1) + P_1(v \cdot \nabla_x P_1 f_1) - P_1 G_1] + L^{-1}P_1(v \cdot \nabla_x P_0 f_1). \quad (4.6)$$

Substituting (4.6) into (4.1)–(4.3), we obtain the system of the compressible Navier-Stokes-Poisson type (NSP) as

$$\partial_t n_1 + \operatorname{div}_x m_1 = 0, \quad (4.7)$$

$$\partial_t m_1 + \partial_t R_1 + \nabla_x n_1 + \sqrt{\frac{2}{3}} \nabla_x q_1 = \kappa_1(\Delta_x m_1 + \frac{1}{3} \nabla_x \operatorname{div}_x m_1) + n_2 \nabla_x \Phi + R_2, \quad (4.8)$$

$$\partial_t q_1 + \partial_t R_3 + \sqrt{\frac{2}{3}} \operatorname{div}_x m_1 = \kappa_2 \Delta_x q_1 + \sqrt{\frac{2}{3}} \nabla_x \Phi \cdot m_2 + R_4, \quad (4.9)$$

where the viscosity coefficients $\kappa_1, \kappa_2 > 0$ and the remainder terms R_1, R_2, R_3, R_4 are defined by

$$\begin{aligned} \kappa_1 &= -(L^{-1}P_1(v_1 \chi_2), v_1 \chi_2), \quad \kappa_2 = -(L^{-1}P_1(v_1 \chi_4), v_1 \chi_4), \\ R_1 &= (v \cdot \nabla_x L^{-1}P_1 f_1, v \sqrt{M}), \quad R_2 = -(v \cdot \nabla_x L^{-1}(P_1(v \cdot \nabla_x P_1 f_1) - P_1 G_1), v \sqrt{M}), \\ R_3 &= (v \cdot \nabla_x L^{-1}P_1 f_1, \chi_4), \quad R_4 = -(v \cdot \nabla_x L^{-1}(P_1(v \cdot \nabla_x P_1 f_1) - P_1 G_1), \chi_4). \end{aligned}$$

By taking inner product between \sqrt{M} and (2.15), we obtain

$$\partial_t n_2 + \operatorname{div}_x m_2 = 0, \quad (4.10)$$

Taking the microscopic projection P_r to (2.15), we have

$$\partial_t(P_r f_2) + P_r(v \cdot \nabla_x P_r f_2) - v \sqrt{M} \cdot \nabla_x \Phi - L_1(P_r f_2) = -P_r(v \cdot \nabla_x P_d f_2) + P_r G_2, \quad (4.11)$$

where the nonlinear term G_2 is denoted by

$$G_2 = \frac{1}{2}(v \cdot \nabla_x \Phi) f_1 - \nabla_x \Phi \cdot \nabla_v f_1 + \Gamma(f_2, f_1). \quad (4.12)$$

By (4.11), we can express the microscopic part $P_r f_2$ as

$$P_r f_2 = L_1^{-1}[\partial_t(P_r f_2) + P_r(v \cdot \nabla_x P_r f_2) - P_r G_2] + L_1^{-1}P_r(v \cdot \nabla_x P_d f_2) - L_1^{-1}(v \sqrt{M} \cdot \nabla_x \Phi). \quad (4.13)$$

Substituting (4.13) into (4.10), we obtain

$$\partial_t n_2 + \partial_t \operatorname{div}_x R_5 + \kappa_3 n_2 - \kappa_3 \Delta_x n_2 = -\operatorname{div}_x((P_r(v \cdot \nabla_x P_r f_2) - P_r G_2), v \sqrt{M}), \quad (4.14)$$

where the viscosity coefficient $\kappa_3 > 0$ and the term R_5 are defined by

$$\kappa_3 = -(L_1^{-1} \chi_1, \chi_1), \quad R_5 = (L_1^{-1} P_r f_2, v \sqrt{M}). \quad (4.15)$$

Lemma 4.1 ([4, 7]). *It holds that*

$$\|\nu^k \partial_v^\beta \Gamma(f, g)\|_{L_v^2} \leq C \sum_{\beta_1 + \beta_2 \leq \beta} (\|\partial_v^{\beta_1} f\|_{L_v^2} \|\nu^{k+1} \partial_v^{\beta_2} g\|_{L_v^2} + \|\nu^{k+1} \partial_v^{\beta_1} f\|_{L_v^2} \|\partial_v^{\beta_2} g\|_{L_v^2}), \quad (4.16)$$

for $k \geq -1$, and

$$\|\Gamma(f, g)\|_{L^{2,1}} \leq C(\|f\|_{L_{x,v}^2} \|\nu g\|_{L_{x,v}^2} + \|\nu f\|_{L_{x,v}^2} \|g\|_{L_{x,v}^2}). \quad (4.17)$$

Lemma 4.2 (Macroscopic dissipation). *Let (n_1, m_1, q_1) and n_2 be the strong solutions to (4.7)–(4.9) and (4.14) respectively. Then, there is a constant $p_0 > 0$ so that*

$$\begin{aligned} & \frac{d}{dt} \sum_{k \leq |\alpha| \leq N-1} p_0 \left(\|\partial_x^\alpha(n_1, m_1, q_1)\|_{L_x^2}^2 + 2 \int_{\mathbb{R}^3} \partial_x^\alpha R_1 \partial_x^\alpha m_1 dx + 2 \int_{\mathbb{R}^3} \partial_x^\alpha R_3 \partial_x^\alpha q_1 dx \right) \\ & + \frac{d}{dt} \sum_{k \leq |\alpha| \leq N-1} 4 \int_{\mathbb{R}^3} \partial_x^\alpha m_1 \partial_x^\alpha \nabla_x n_1 dx + \sum_{k \leq |\alpha| \leq N-1} \|\partial_x^\alpha \nabla_x(n_1, m_1, q_1)\|_{L_x^2}^2 \\ & \leq C \sqrt{E(f_1, f_2)} D(f_1, f_2) + C \sum_{k \leq |\alpha| \leq N-1} \|\partial_x^\alpha \nabla_x P_1 f_1\|_{L_{x,v}^2}^2, \end{aligned} \quad (4.18)$$

$$\begin{aligned} & \frac{d}{dt} \sum_{k \leq |\alpha| \leq N-1} (\|\partial_x^\alpha n_2\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2 + 2 \int_{\mathbb{R}^3} \partial_x^\alpha \operatorname{div}_x R_5 \partial_x^\alpha n_2 dx + 2 \int_{\mathbb{R}^3} \partial_x^\alpha R_5 \partial_x^\alpha \nabla_x \Phi dx) \\ & + \kappa_3 \sum_{k \leq |\alpha| \leq N-1} (\|\partial_x^\alpha n_2\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x n_2\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2) \\ & \leq C E(f_1, f_2) D(f_1, f_2) + C \sum_{k \leq |\alpha| \leq N-1} (\|\partial_x^\alpha P_r f_2\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x P_r f_2\|_{L_{x,v}^2}^2), \end{aligned} \quad (4.19)$$

with $0 \leq k \leq N-1$.

Proof. First of all, we prove (4.18). Taking the inner product between $\partial_x^\alpha m_1$ and ∂_x^α (4.8) with $|\alpha| \leq N-1$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_x^\alpha m_1\|_{L_x^2}^2 + \int_{\mathbb{R}^3} \partial_x^\alpha \partial_t R_1 \partial_x^\alpha m_1 dx + \int_{\mathbb{R}^3} \partial_x^\alpha \nabla_x n_1 \partial_x^\alpha m_1 dx + \sqrt{\frac{2}{3}} \int_{\mathbb{R}^3} \partial_x^\alpha \nabla_x q_1 \partial_x^\alpha m_1 dx \\ & + \kappa_1 (\|\partial_x^\alpha \nabla_x m_1\|_{L_x^2}^2 + \frac{1}{3} \|\partial_x^\alpha \operatorname{div}_x m_1\|_{L_x^2}^2) \\ & = \int_{\mathbb{R}^3} \partial_x^\alpha (n_2 \nabla_x \Phi) \partial_x^\alpha m_1 dx + \int_{\mathbb{R}^3} \partial_x^\alpha R_1 \partial_x^\alpha m_1 dx. \end{aligned} \quad (4.20)$$

For the second and third terms in the left hand side of (4.20), we have

$$\begin{aligned} \int_{\mathbb{R}^3} \partial_x^\alpha \partial_t R_1 \partial_x^\alpha m_1 dx & = \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha R_1 \partial_x^\alpha m_1 dx \\ & - \int_{\mathbb{R}^3} \partial_x^\alpha R_1 \partial_x^\alpha [-\nabla_x n_1 - \sqrt{\frac{2}{3}} \nabla_x q_1 + n_2 \nabla_x \Phi - (v \cdot \nabla_x P_1 f_1, v \sqrt{M})] dx \\ & \geq \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha R_1 \partial_x^\alpha m_1 dx - \epsilon (\|\partial_x^\alpha \nabla_x n_1\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x q_1\|_{L_x^2}^2) \\ & - C \sqrt{E(f_1, f_2)} D(f_1, f_2) - \frac{C}{\epsilon} \|\partial_x^\alpha \nabla_x P_1 f_1\|_{L_x^2}^2, \end{aligned} \quad (4.21)$$

and

$$\int_{\mathbb{R}^3} \partial_x^\alpha \nabla_x n_1 \partial_x^\alpha m_1 dx = - \int_{\mathbb{R}^3} \partial_x^\alpha n \partial_x^\alpha \operatorname{div}_x m dx = \int_{\mathbb{R}^3} \partial_x^\alpha n_1 \partial_x^\alpha \partial_t n_1 dx = \frac{1}{2} \frac{d}{dt} \|\partial_x^\alpha n_1\|_{L_x^2}^2. \quad (4.22)$$

The first term in the right hand side of (4.20) are bounded by $C \sqrt{E(f_1, f_2)} D(f_1, f_2)$. The second term can be estimated by

$$\begin{aligned} \int_{\mathbb{R}^3} \partial_x^\alpha R_2 \partial_x^\alpha m_1 dx & \leq C \|\partial_x^\alpha \nabla_x P_1 f_1\|_{L_{x,v}^2} \|\partial_x^\alpha \nabla_x m_1\|_{L_x^2} \\ & + C (\|\partial_x^\alpha (\nabla_x \Phi f_2)\|_{L_{x,v}^2} + \|w^{-\frac{1}{2}} \partial_x^\alpha \Gamma(f_1, f_1)\|_{L_{x,v}^2}) \|\partial_x^\alpha \nabla_x m_1\|_{L_x^2} \\ & \leq \frac{\kappa_1}{2} \|\partial_x^\alpha \nabla_x m_1\|_{L_x^2}^2 + C \|\partial_x^\alpha \nabla_x P_1 f_1\|_{L_{x,v}^2}^2 + C \sqrt{E(f_1, f_2)} D(f_1, f_2), \end{aligned} \quad (4.23)$$

where we make use of Lemma 4.1 to obtain

$$\|w^{-\frac{1}{2}} \partial_x^\alpha \Gamma(f_1, f_1)\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha (\nabla_x \Phi f_2)\|_{L_{x,v}^2}^2$$

$$\begin{aligned}
&\leq C\|f_1\|_{L_v^2(H_x^N)}^2\|w^{\frac{1}{2}}\nabla_x f_1\|_{L_v^2(H_x^{N-1})}^2 + C\|\nabla_x \Phi\|_{H_x^N}^2\|\nabla_x f_2\|_{L_v^2(H_x^{N-1})}^2 \\
&\leq CE(f_1, f_2)D(f_1, f_2).
\end{aligned} \tag{4.24}$$

Therefore, it follows from (4.20)–(4.23) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\partial_x^\alpha m_1\|_{L_x^2}^2 + \|\partial_x^\alpha n_1\|_{L_x^2}^2) + \frac{d}{dt}\int_{\mathbb{R}^3}\partial_x^\alpha R_1\partial_x^\alpha m_1 dx + \sqrt{\frac{2}{3}}\int_{\mathbb{R}^3}\partial_x^\alpha \nabla_x q_1\partial_x^\alpha m_1 dx \\
&\quad + \frac{\kappa_1}{2}(\|\partial_x^\alpha \nabla_x m_1\|_{L_x^2}^2 + \frac{1}{3}\|\partial_x^\alpha \operatorname{div}_x m_1\|_{L_x^2}^2) \\
&\leq C\sqrt{E(f_1, f_2)}D(f_1, f_2) + \frac{C}{\epsilon}\|\partial_x^\alpha \nabla_x P_1 f_1\|_{L_{x,v}^2}^2 + \epsilon(\|\partial_x^\alpha \nabla_x n_1\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x q_1\|_{L_x^2}^2),
\end{aligned} \tag{4.25}$$

Similarly, taking the inner product between $\partial_x^\alpha q_1$ and ∂_x^α (4.9) with $|\alpha| \leq N-1$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha q_1\|_{L_x^2}^2 + \frac{d}{dt}\int_{\mathbb{R}^3}\partial_x^\alpha R_3\partial_x^\alpha q_1 dx + \sqrt{\frac{2}{3}}\int_{\mathbb{R}^3}\partial_x^\alpha \operatorname{div}_x m_1\partial_x^\alpha q_1 dx + \frac{\kappa_2}{2}\|\partial_x^\alpha \nabla_x q_1\|_{L_x^2}^2 \\
&\leq C\sqrt{E(f_1, f_2)}D(f_1, f_2) + \frac{C}{\epsilon}\|\partial_x^\alpha \nabla_x P_1 f_1\|_{L_{x,v}^2}^2 + \epsilon\|\partial_x^\alpha \nabla_x m_1\|_{L_x^2}^2.
\end{aligned} \tag{4.26}$$

Again, taking the inner product between $\partial_x^\alpha \nabla_x n_1$ and ∂_x^α (4.2) with $|\alpha| \leq N-1$ to get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\partial_x^\alpha m_1\partial_x^\alpha \nabla_x n_1 dx + \frac{1}{2}\|\partial_x^\alpha \nabla_x n_1\|_{L_x^2}^2 \\
&\leq C\sqrt{E(f_1, f_2)}D(f_1, f_2) + \|\partial_x^\alpha \operatorname{div}_x m_1\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x q_1\|_{L_x^2}^2 + C\|\partial_x^\alpha \nabla_x P_1 f_1\|_{L_{x,v}^2}^2.
\end{aligned} \tag{4.27}$$

Taking the summation of $p_0 \sum_{k \leq |\alpha| \leq N-1} [(4.25) + (4.26)] + 4 \sum_{k \leq |\alpha| \leq N-1} (4.27)$ with $p_0 > 0$ large enough, $\epsilon > 0$ small enough and $0 \leq k \leq N-1$, we obtain (4.18).

Next, we turn to show (4.19). Taking the inner product between $\partial_x^\alpha n_2$ and ∂_x^α (4.14) with $|\alpha| \leq N-1$, we have

$$\begin{aligned}
&\frac{d}{dt}\|\partial_x^\alpha n_2\|_{L_x^2}^2 + 2\frac{d}{dt}\int_{\mathbb{R}^3}\partial_x^\alpha \operatorname{div}_x R_5\partial_x^\alpha n_2 dx + \kappa_3\|\partial_x^\alpha n_2\|_{L_x^2}^2 + \kappa_3\|\partial_x^\alpha \nabla_x n_2\|_{L_x^2}^2 \\
&\leq C\|\partial_x^\alpha \nabla_x P_r f_2\|_{L_{x,v}^2}^2 + CE(f_1, f_2)D(f_1, f_2).
\end{aligned} \tag{4.28}$$

Similarly, taking the inner product between $-\partial_x^\alpha \Phi$ and ∂_x^α (4.14) with $|\alpha| \leq N-1$ we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2 + 2\frac{d}{dt}\int_{\mathbb{R}^3}\partial_x^\alpha R_5\partial_x^\alpha \nabla_x \Phi dx + \kappa_3\|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2 + \kappa_3\|\partial_x^\alpha n_2\|_{L_x^2}^2 \\
&\leq C(\|\partial_x^\alpha P_r f_2\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x P_r f_2\|_{L_{x,v}^2}^2) + CE(f_1, f_2)D(f_1, f_2).
\end{aligned} \tag{4.29}$$

Taking the summation of $\sum_{k \leq |\alpha| \leq N-1} [(4.28) + (4.29)]$ with $0 \leq k \leq N-1$, we obtain (4.19). \square

In the following, we shall estimate the microscopic terms to enclose the energy inequality of the solution f to bVPB system (2.14)–(2.16).

Lemma 4.3 (Microscopic dissipation). *Let (f_1, f_2, Φ) be a strong solution to bVPB system (2.14)–(2.16). Then, there are constants $p_k > 0$, $1 \leq k \leq N$ so that*

$$\frac{1}{2}\frac{d}{dt}(\|(f_1, f_2)\|_{L_{x,v}^2}^2 + \|\nabla_x \Phi\|_{L_x^2}^2) + \mu\|w^{\frac{1}{2}}(P_1 f_1, P_r f_2)\|_{L_{x,v}^2}^2 \leq C\sqrt{E(f_1, f_2)}D(f_1, f_2), \tag{4.30}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \sum_{1 \leq |\alpha| \leq N} (\|\partial_x^\alpha (f_1, f_2)\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2) + \mu \sum_{1 \leq |\alpha| \leq N} \|w^{\frac{1}{2}}\partial_x^\alpha (P_1 f_1, P_r f_2)\|_{L_{x,v}^2}^2 \\
&\leq C\sqrt{E(f_1, f_2)}D(f_1, f_2),
\end{aligned} \tag{4.31}$$

$$\frac{d}{dt} \sum_{1 \leq k \leq N} p_k \sum_{\substack{|\beta|=k \\ |\alpha|+|\beta| \leq N}} \|\partial_x^\alpha \partial_v^\beta (P_1 f_1, P_r f_2)\|_{L_{x,v}^2}^2 + \mu \sum_{1 \leq k \leq N} p_k \sum_{\substack{|\beta|=k \\ |\alpha|+|\beta| \leq N}} \|w^{\frac{1}{2}}\partial_x^\alpha \partial_v^\beta (P_1 f_1, P_r f_2)\|_{L_{x,v}^2}^2$$

$$\leq C \sum_{|\alpha| \leq N-1} (\|\partial_x^\alpha \nabla_x(f_1, f_2)\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2) + C\sqrt{E(f_1, f_2)}D(f_1, f_2). \quad (4.32)$$

Proof. Taking inner product between $\partial_x^\alpha f_1$ and $\partial_x^\alpha (2.14)$ with $|\alpha| \leq N$ ($N \geq 4$), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_x^\alpha f_1\|_{L_{x,v}^2}^2 - \int_{\mathbb{R}^3} (L \partial_x^\alpha f_1) \partial_x^\alpha f_1 dx dv \\ &= \frac{1}{2} \int_{\mathbb{R}^6} \partial_x^\alpha (v \cdot \nabla_x \Phi f_2) \partial_x^\alpha f_1 dx dv - \int_{\mathbb{R}^6} \partial_x^\alpha (\nabla_x \Phi \cdot \nabla_v f_2) \partial_x^\alpha f_1 dx dv + \int_{\mathbb{R}^6} \partial_x^\alpha \Gamma(f_1, f_1) \partial_x^\alpha f_1 dx dv \\ &=: I_1 + I_2 + I_3. \end{aligned} \quad (4.33)$$

For I_1 , it holds

$$\begin{aligned} I_1 &\leq C \sum_{1 \leq |\alpha'| \leq |\alpha|-1} \int_{\mathbb{R}^3} |v| \|\partial_x^{\alpha'} \nabla_x \Phi\|_{L_x^3} \|\partial_x^{\alpha-\alpha'} f_2\|_{L_x^6} \|\partial_x^\alpha f_1\|_{L_x^2} dv \\ &\quad + C \int_{\mathbb{R}^3} |v| (\|\nabla_x \Phi\|_{L_x^\infty} \|\partial_x^\alpha f_2\|_{L_x^2} + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2} \|f_2\|_{L_x^\infty}) \|\partial_x^\alpha f_1\|_{L_x^2} dv \\ &\leq C \|\nabla_x \Phi\|_{H_x^N} \|w^{\frac{1}{2}} \nabla_x f_2\|_{L_v^2(H_x^{N-1})} \|w^{\frac{1}{2}} \partial_x^\alpha f_1\|_{L_{x,v}^2} \leq C\sqrt{E(f_1, f_2)}D(f_1, f_2), \end{aligned} \quad (4.34)$$

for $|\alpha| \geq 1$, and

$$I_1 \leq \int_{\mathbb{R}^3} |v| \|\nabla_x \Phi\|_{L_x^3} \|f_2\|_{L_x^2} \|f_1\|_{L_x^6} dv \leq C\sqrt{E(f_1, f_2)}D(f_1, f_2), \quad (4.35)$$

for $|\alpha| = 0$. For I_2 , it holds

$$\begin{aligned} I_2 &\leq C \sum_{|\alpha'| \leq N/2} \int_{\mathbb{R}^3} \|\partial_x^{\alpha'} \nabla_x \Phi\|_{L_x^\infty} \|\partial_x^{\alpha-\alpha'} \nabla_v f_2\|_{L_x^2} \|\partial_x^\alpha f_1\|_{L_x^2} dv \\ &\quad + C \sum_{|\alpha'| \geq N/2} \int_{\mathbb{R}^3} \|\partial_x^{\alpha'} \nabla_x \Phi\|_{L_x^2} \|\partial_x^{\alpha-\alpha'} \nabla_v f_2\|_{L_x^\infty} \|\partial_x^\alpha f_1\|_{L_x^2} dv - \int_{\mathbb{R}^6} \nabla_x \Phi \partial_x^\alpha \nabla_v f_2 \partial_x^\alpha f_1 dx dv \\ &\leq C\sqrt{E(f_1, f_2)}D(f_1, f_2) - \int_{\mathbb{R}^6} \nabla_x \Phi \partial_x^\alpha \nabla_v f_2 \partial_x^\alpha f_1 dx dv. \end{aligned} \quad (4.36)$$

For I_3 , by (4.24) we obtain

$$I_3 \leq \|w^{-\frac{1}{2}} \partial_x^\alpha \Gamma(f_1, f_1)\|_{L_{x,v}^2} \|w^{\frac{1}{2}} \partial_x^\alpha P_1 f_1\|_{L_{x,v}^2} \leq C\sqrt{E(f_1, f_2)}D(f_1, f_2). \quad (4.37)$$

Therefore, it follows from (4.33)–(4.37) that

$$\frac{1}{2} \frac{d}{dt} \|\partial_x^\alpha f_1\|_{L_{x,v}^2}^2 + \mu \|w^{\frac{1}{2}} \partial_x^\alpha P_1 f_1\|_{L_{x,v}^2}^2 \leq C\sqrt{E(f_1, f_2)}D(f_1, f_2) - \int_{\mathbb{R}^6} \nabla_x \Phi \partial_x^\alpha \nabla_v f_2 \partial_x^\alpha f_1 dx dv. \quad (4.38)$$

Similarly, taking inner product between $\partial_x^\alpha f_2$ and $\partial_x^\alpha (2.15)$ with $|\alpha| \leq N$ ($N \geq 4$) we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\partial_x^\alpha f_2\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2) - \int_{\mathbb{R}^3} (L_1 \partial_x^\alpha f_2) \partial_x^\alpha f_2 dx dv \\ &\leq C\sqrt{E(f_1, f_2)}D(f_1, f_2) - \int_{\mathbb{R}^6} \nabla_x \Phi \partial_x^\alpha \nabla_v f_1 \partial_x^\alpha f_2 dx dv. \end{aligned} \quad (4.39)$$

Taking the summation (4.38)+(4.39) for $|\alpha| = 0$ and $\sum_{1 \leq |\alpha| \leq N} [(4.38) + (4.39)]$, we obtain (4.30) and (4.31) respectively.

In order to enclose the energy inequality, we need to estimate the terms $\partial_x^\alpha \nabla_v f$ with $|\alpha| \leq N-1$. To this end, we rewrite (4.6) and (4.13) as

$$\partial_t(P_1 f_1) + v \cdot \nabla_x P_1 f_1 + \nabla_x \Phi \cdot \nabla_v P_r f_2 - L(P_1 f_1)$$

$$= \Gamma(f_1, f_1) + \frac{1}{2}v \cdot \nabla_x \Phi P_r f_2 + P_0(v \cdot \nabla_x P_1 f_1 - \frac{1}{2}v \cdot \nabla_x \Phi P_r f_2 + \nabla_x \Phi \cdot \nabla_v P_r f_2) - P_1(v \cdot \nabla_x P_0 f_1), \quad (4.40)$$

and

$$\begin{aligned} & \partial_t(P_r f_2) + v \cdot \nabla_x P_r f_2 - v \sqrt{M} \cdot \nabla_x \Phi + \nabla_x \Phi \cdot \nabla_v P_1 f_1 + L_1(P_r f_2) \\ &= \Gamma(f_2, f_1) + \frac{1}{2}v \cdot \nabla_x \Phi P_1 f_1 + P_d(v \cdot \nabla_x P_r f_2) - (v \cdot \nabla_x P_d f_2 - \frac{1}{2}v \cdot \nabla_x \Phi P_0 f_1 + \nabla_x \Phi \cdot \nabla_v P_0 f_1). \end{aligned} \quad (4.41)$$

Let $1 \leq k \leq N$, and choose α, β with $|\beta| = k$ and $|\alpha| + |\beta| \leq N$. Taking inner product between $\partial_x^\alpha \partial_v^\beta P_1 f_1$ and (4.40) and between $\partial_x^\alpha \partial_v^\beta P_r f_2$ and (4.41) respectively, and then taking summation of the resulted equations, we have

$$\begin{aligned} & \frac{d}{dt} \sum_{\substack{|\beta|=k \\ |\alpha|+|\beta| \leq N}} \|\partial_x^\alpha \partial_v^\beta (P_1 f_1, P_r f_2)\|_{L_{x,v}^2}^2 + \mu \sum_{\substack{|\beta|=k \\ |\alpha|+|\beta| \leq N}} \|w^{\frac{1}{2}} \partial_x^\alpha \partial_v^\beta (P_1 f_1, P_r f_2)\|_{L_{x,v}^2}^2 \\ & \leq C \sum_{|\alpha| \leq N-k} (\|\partial_x^\alpha \nabla_x (P_0 f_1, P_d f_2)\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x (P_1 f_1, P_r f_2)\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2) \\ & \quad + C_k \sum_{\substack{|\beta| \leq k-1 \\ |\alpha|+|\beta| \leq N}} \|\partial_x^\alpha \partial_v^\beta (P_1 f_1, P_r f_2)\|_{L_{x,v}^2}^2 + C \sqrt{E(f_1, f_2)} D(f_1, f_2), \end{aligned} \quad (4.42)$$

Then taking summation $\sum_{1 \leq k \leq N} p_k$ (4.42) with constants p_k chosen by

$$\mu p_k \geq 2 \sum_{1 \leq j \leq N-k} p_{k+j} C_{k+j}, \quad 1 \leq k \leq N-1, \quad p_N = 1,$$

we obtain (4.32). The proof of the lemma is completed. \square

With the help of Lemma 4.2–4.3, we have the following result.

Lemma 4.4. *Let $N \geq 4$. Then, there are two equivalent energy functionals $E_0^f(\cdot) \sim E(\cdot)$ and $H_0^f(\cdot) \sim H(\cdot)$ such that the following holds. If $E(f_{1,0}, f_{2,0})$ is sufficiently small, then the Cauchy problem (2.14)–(2.17) of the bVPB system admits a unique global solution $(f_1, f_2)(x, v, t)$ satisfying*

$$\frac{d}{dt} E_0^f(f_1, f_2)(t) + \mu D(f_1, f_2)(t) \leq 0, \quad (4.43)$$

$$\frac{d}{dt} H_0^f(f_1, f_2)(t) + \mu D(f_1, f_2)(t) \leq C \|\nabla_x(n_1, m_1, q_1)\|_{L_x^2}^2. \quad (4.44)$$

Proof. Assume that

$$E(f_1, f_2)(t) \leq \delta$$

for $\delta > 0$ small.

Taking the summation of $A_1[(4.18) + (4.19)] + A_2[(4.30) + (4.31)] + (4.32)$ with $A_2 > C_0 A_1 > 0$ large enough and taking $k = 0$ in (4.18) and (4.19), we prove (4.43).

Taking $|\alpha| = 0$ in (4.39) and noting that the terms in the right hand side are bounded by $\sqrt{E(f_1, f_2)} D(f_1, f_2)$, we have

$$\frac{1}{2} \frac{d}{dt} (\|f_2\|_{L_{x,v}^2}^2 + \|\nabla_x \Phi\|_{L_x^2}^2) + \mu \|w^{\frac{1}{2}} P_r f_2\|_{L_{x,v}^2}^2 \leq \sqrt{E(f_1, f_2)} D(f_1, f_2). \quad (4.45)$$

Taking the inner product between (4.40) and $P_1 f_1$, we have

$$\frac{d}{dt} \|P_1 f_1\|_{L_{x,v}^2}^2 + \|w^{\frac{1}{2}} P_1 f_1\|_{L_{x,v}^2}^2 \leq C \|\nabla_x P_0 f_1\|_{L_{x,v}^2}^2 + E(f_1, f_2) D(f_1, f_2). \quad (4.46)$$

Taking the summation of $A_1[(4.18) + (4.19)] + A_2[(4.45) + (4.46) + (4.31)] + (4.32)$ with $A_2 > C_0 A_1 > 0$ large enough and taking $k = 1$ in (4.18) and $k = 0$ in (4.19), we prove (4.44). \square

Repeating the proof of Lemmas 4.2–4.3, we can show

Lemma 4.5. *Let $N \geq 4$. There are the equivalent energy functionals $E_1^f(\cdot) \sim E_1(\cdot)$, $H_1^f(\cdot) \sim H_1(\cdot)$ such that if $E_1(f_{1,0}, f_{2,0})$ is sufficiently small, then the solution $(f_1, f_2)(x, v, t)$ to the bVPB system (2.14)–(2.17) satisfies*

$$\frac{d}{dt} E_1^f(f_1, f_2)(t) + \mu D_1(f_1, f_2)(t) \leq 0, \quad (4.47)$$

$$\frac{d}{dt} H_1^f(f_1, f_2)(t) + \mu D_1(f_1, f_2)(t) \leq C \|\nabla_x(n_1, m_1, q_1)\|_{L_x^2}^2. \quad (4.48)$$

4.2 Convergence rates

Theorem 4.6. *Assume that $(f_{1,0}, f_{2,0}) \in H_w^N \cap L^{2,1}$ for $N \geq 4$ and $\|(f_{1,0}, f_{2,0})\|_{H_w^N \cap L^{2,1}} \leq \delta_0$ for a constant $\delta_0 > 0$ small enough. Then, the global solution $(f_1, f_2)(x, v, t)$ to the bVPB system (2.14)–(2.17) satisfies*

$$\|\partial_x^k f_2(t)\|_{L_{x,v}^2} + \|\partial_x^k \nabla_x \Phi(t)\|_{L_x^2} \leq C \delta_0 e^{-dt}, \quad t > 0, \quad (4.49)$$

for $k = 0, 1$ and $d > 0$ a constant.

Proof. Let (f_1, f_2) be a solution to the Cauchy problem (2.14)–(2.17) for $t > 0$. Firstly, we prove that there is a constant $K_2 > 0$ and energy functionals $E_2(f_2), D_2(f_2)$ of f_2 defined by

$$\begin{aligned} E_2(f_2) &= K_2 \sum_{|\alpha| \leq 1} (\|\partial_x^\alpha f_2\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2) + \|n_2\|_{L_x^2}^2 + \|\nabla_x \Phi\|_{L_x^2}^2 \\ &\quad + 2 \int_{\mathbb{R}^3} \operatorname{div}_x R_5 n_2 dx + 2 \int_{\mathbb{R}^3} R_5 \nabla_x \Phi dx, \\ D_2(f_2) &= \sum_{|\alpha| \leq 1} \|w^{\frac{1}{2}} \partial_x^\alpha P_r f_2\|_{L_{x,v}^2}^2 + \|n_2\|_{L_x^2}^2 + \|\nabla_x n_2\|_{L_x^2}^2 + \|\nabla_x \Phi\|_{L_x^2}^2, \end{aligned}$$

such that

$$\frac{d}{dt} E_2(f_2(t)) + \kappa_3 D_2(f_2(t)) \leq 0 \quad \text{and} \quad E_2(f_2) \leq C D_2(f_2). \quad (4.50)$$

Indeed, taking the inner product between n_2 and (4.7) and using Cauchy-Schwarz inequality and Sobolev embedding, we obtain

$$\begin{aligned} &\frac{d}{dt} (\|n_2\|_{L_x^2}^2 + 2 \int_{\mathbb{R}^3} \operatorname{div}_x R_5 n_2 dx) + \kappa_3 (\|n_2\|_{L_x^2}^2 + \|\nabla_x n_2\|_{L_x^2}^2) \\ &\leq C \|\nabla_x P_r f_2\|_{L_{x,v}^2}^2 + C E_1(f_1, f_2) D_2(f_2). \end{aligned} \quad (4.51)$$

Similarly, taking the inner product between $-\Phi$ and (4.7) we obtain

$$\begin{aligned} &\frac{d}{dt} (\|\nabla_x \Phi\|_{L_x^2}^2 + 2 \int_{\mathbb{R}^3} R_5 \nabla_x \Phi dx) + \kappa_3 (\|\nabla_x \Phi\|_{L_x^2}^2 + \|n_2\|_{L_x^2}^2) \\ &\leq C \|P_r f_2\|_{L_{x,v}^2}^2 + C E_1(f_1, f_2) D_2(f_2). \end{aligned} \quad (4.52)$$

Taking the inner product between f_2 and (2.15), we have

$$\frac{1}{2} \frac{d}{dt} (\|f_2\|_{L_{x,v}^2}^2 + \|\nabla_x \Phi\|_{L_x^2}^2) - \int_{\mathbb{R}^3} (L_1 f_2) f_2 dx dv \leq C \sqrt{E_1(f_1, f_2)} D_2(f_2). \quad (4.53)$$

Again, taking the inner product between $\partial_x^\alpha f_2$ and ∂_x^α (2.15) with $|\alpha| = 1$ to get

$$\frac{1}{2} \frac{d}{dt} (\|\partial_x^\alpha f_2\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2) - \int_{\mathbb{R}^3} (L_1 \partial_x^\alpha f_2) \partial_x^\alpha f_2 dx dv \leq C \sqrt{E_1(f_1, f_2)} D_2(f_2). \quad (4.54)$$

Making the summation of $K_2[(4.53) + (4.54)] + (4.51) + (4.52)$ for a sufficiently large constant $K_2 > 0$, we obtain (4.50) for $E_1(f_1, f_2) > 0$ small enough due to Lemma 4.5. Then (4.49) follows from (4.50) by Gronwall's inequality. \square

Theorem 4.7. Assume that $(f_{1,0}, f_{2,0}) \in H_w^N \cap L^{2,1}$ for $N \geq 4$ and $\|(f_{1,0}, f_{2,0})\|_{H_w^N \cap L^{2,1}} \leq \delta_0$ for a constant $\delta_0 > 0$ small enough. Then, the global solution $(f_1, f_2)(x, v, t)$ to the bVPB system (2.14)–(2.17) satisfies

$$\|\partial_x^k(f_1(t), \chi_j)\|_{L_x^2} \leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad j = 0, 1, 2, 3, 4, \quad (4.55)$$

$$\|\partial_x^k P_1 f_1(t)\|_{L_{x,v}^2} \leq C\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \quad (4.56)$$

$$\|(P_1 f_1, P_r f_2)(t)\|_{H_w^N} + \|\nabla_x(P_0 f_1, P_d f_2)(t)\|_{L_v^2(H_x^{N-1})} \leq C\delta_0(1+t)^{-\frac{5}{4}}, \quad (4.57)$$

for $k = 0, 1$.

Proof. Let (f_1, f_2) be a solution to the Cauchy problem (2.14)–(2.17) for $t > 0$. We can represent the solution in terms of the semigroup e^{tE}, e^{tB} as

$$f_1(t) = e^{tE} f_{1,0} + \int_0^t e^{(t-s)E} G_1(s) ds, \quad (4.58)$$

$$f_2(t) = e^{tB} f_{2,0} + \int_0^t e^{(t-s)B} G_2(s) ds, \quad (4.59)$$

where the nonlinear terms G_1 and G_2 are given by (4.5) and (4.12) respectively. For this global solution f_1, f_2 , we define a functional $Q(t)$ for any $t > 0$ as

$$Q(t) = \sup_{0 \leq s \leq t} \sum_{k=0}^1 \left\{ \sum_{j=0}^4 \|\partial_x^k(f_1(s), \chi_j)\|_{L_x^2} (1+s)^{\frac{3}{4}+\frac{k}{2}} + \|\partial_x^k P_1 f_1(s)\| (1+s)^{\frac{5}{4}+\frac{k}{2}} \right. \\ \left. + (\|(P_1 f_1, P_r f_2)(s)\|_{H_w^N} + \|\nabla_x P_0 f_1(s)\|_{L_v^2(H_x^{N-1})} + \|P_d f_2(s)\|_{L_v^2(H_x^N)}) (1+s)^{\frac{5}{4}} \right\}.$$

We claim that it holds under the assumptions of Theorem 4.7 that

$$Q(t) \leq C\delta_0. \quad (4.60)$$

First, let us deal with the time-decay rate of the macroscopic density, momentum and energy of f_1 , which in terms of (4.58) satisfy the following equations

$$(f_1(t), \chi_j) = (e^{tE} f_{1,0}, \chi_j) + \int_0^t (e^{(t-s)E} G_1(s), \chi_j) ds, \quad j = 0, 1, 2, 3, 4. \quad (4.61)$$

By Lemma 4.1 and (2.48), we can estimate the nonlinear term $G_1(s)$ for $0 \leq s \leq t$ in terms of $Q(t)$ as

$$\begin{aligned} \|G_1(s)\|_{L_{x,v}^2} &\leq C\{\|w f_1\|_{L^{2,3}} \|f_1\|_{L^{2,6}} + \|\nabla_x \Phi\|_{L_x^3} (\|w f_2\|_{L^{2,6}} + \|\nabla_v f_2\|_{L^{2,6}})\} \\ &\leq C(1+s)^{-2} Q(t)^2 + C\delta_0 e^{-ds} (1+s)^{-\frac{5}{4}} Q(t), \end{aligned} \quad (4.62)$$

$$\begin{aligned} \|G_1(s)\|_{L^{2,1}} &\leq C\{\|f_1\|_{L_{x,v}^2} \|w f_1\|_{L_{x,v}^2} + \|\nabla_x \Phi\|_{L_x^3} (\|w f_2\|_{L_{x,v}^2} + \|\nabla_v f_1\|_{L_{x,v}^2})\} \\ &\leq C(1+s)^{-\frac{3}{2}} Q(t)^2 + C\delta_0 e^{-ds} (1+s)^{-\frac{3}{4}} Q(t), \end{aligned} \quad (4.63)$$

Then, it follows from (2.36), (4.62) and (4.63) that

$$\begin{aligned} \|(f_1(t), \chi_j)\|_{L_x^2} &\leq C(1+t)^{-\frac{3}{4}} (\|f_{1,0}\|_{L_{x,v}^2} + \|f_{1,0}\|_{L^{2,1}}) \\ &\quad + C \int_0^t (1+t-s)^{-\frac{3}{4}} (\|G_1(s)\|_{L_{x,v}^2} + \|G_1(s)\|_{L^{2,1}}) ds \\ &\leq C\delta_0(1+t)^{-\frac{3}{4}} + C \int_0^t (1+t-s)^{-\frac{3}{4}} [(1+s)^{-\frac{3}{2}} Q(t)^2 + C\delta_0 e^{-ds} (1+s)^{-\frac{3}{4}} Q(t)] ds \\ &\leq C\delta_0(1+t)^{-\frac{3}{4}} + C\delta_0(1+t)^{-\frac{3}{4}} Q(t) + C(1+t)^{-\frac{3}{4}} Q(t)^2. \end{aligned} \quad (4.64)$$

Similarly, we have

$$\|(\nabla_x f_1(t), \chi_j)\|_{L_x^2} \leq C(1+t)^{-\frac{5}{4}} (\|\nabla_x f_{1,0}\|_{L_{x,v}^2} + \|f_{1,0}\|_{L^{2,1}})$$

$$\begin{aligned}
& + C \int_0^t (1+t-s)^{-\frac{5}{4}} (\|\nabla_x G_1(s)\|_{L_{x,v}^2} + \|G_1(s)\|_{L^{2,1}}) ds \\
& \leq C\delta_0(1+t)^{-\frac{5}{4}} + C \int_0^t (1+t-s)^{-\frac{5}{4}} [(1+s)^{-\frac{3}{2}} Q(t)^2 + C\delta_0 e^{-ds} (1+s)^{-\frac{3}{4}} Q(t)] ds \\
& \leq C\delta_0(1+t)^{-\frac{5}{4}} + C\delta_0(1+t)^{-\frac{5}{4}} Q(t) + C(1+t)^{-\frac{5}{4}} Q(t)^2,
\end{aligned} \tag{4.65}$$

where we use

$$\|\nabla_x G_1(s)\|_{L_{x,v}^2} + \|G_1(s)\|_{L^{2,1}} \leq C(1+s)^{-2} Q(t)^2 + C\delta_0 e^{-ds} (1+s)^{-\frac{5}{4}} Q(t).$$

Second, we estimate the microscopic part $P_1 f_1(t)$ as below. Since

$$P_1 f_1(t) = P_1(e^{tE} f_{1,0}) + \int_0^t P_1(e^{(t-s)E} G_1(s)) ds,$$

it follows from (2.37), (4.62) and (4.63) that

$$\begin{aligned}
\|P_1 f_1(t)\|_{L_{x,v}^2} & \leq C(1+t)^{-\frac{5}{4}} (\|f_{1,0}\|_{L_{x,v}^2} + \|f_{1,0}\|_{L^{2,1}}) \\
& + C \int_0^t (1+t-s)^{-\frac{5}{4}} (\|G_1(s)\|_{L_{x,v}^2} + \|G_1(s)\|_{L^{2,1}}) ds \\
& \leq C\delta_0(1+t)^{-\frac{5}{4}} + C\delta_0(1+t)^{-\frac{5}{4}} Q(t) + C(1+t)^{-\frac{5}{4}} Q(t)^2,
\end{aligned} \tag{4.66}$$

and

$$\begin{aligned}
\|\nabla_x P_1 f_1(t)\|_{L_{x,v}^2} & \leq C(1+t)^{-\frac{7}{4}} (\|\nabla_x f_{1,0}\|_{L_{x,v}^2} + \|f_{1,0}\|_{L^{2,1}}) \\
& + C \int_0^{t/2} (1+t-s)^{-\frac{7}{4}} (\|\nabla_x G_1(s)\|_{L_{x,v}^2} + \|G_1(s)\|_{L^{2,1}}) ds \\
& + C \int_{t/2}^t (1+t-s)^{-\frac{5}{4}} (\|\nabla_x G_1(s)\|_{L_{x,v}^2} + \|\nabla_x G_1(s)\|_{L^{2,1}}) ds \\
& \leq C\delta_0(1+t)^{-\frac{7}{4}} + C\delta_0(1+t)^{-\frac{7}{4}} Q(t) + C(1+t)^{-\frac{7}{4}} Q(t)^2,
\end{aligned} \tag{4.67}$$

where we have used

$$\|\nabla_x G_1(s)\|_{L_{x,v}^2} + \|\nabla_x G_1(s)\|_{L^{2,1}} \leq C(1+s)^{-2} Q(t)^2 + C\delta_0 e^{-ds} (1+s)^{-\frac{5}{4}} Q(t).$$

Next, we estimate the higher order terms as below. By (4.48) and $cH_1^f(f_1, f_2) \leq D_1(f_1, f_2)$ with $c > 0$, we have

$$\begin{aligned}
H_1^f(f_1, f_2)(t) & \leq e^{-c\mu t} H_1^f(f_{1,0}, f_{2,0}) + \int_0^t e^{-c\mu(t-s)} \|\nabla_x(n_1, m_1, q_1)(s)\|_{L_x^2}^2 ds \\
& \leq C\delta_0^2 e^{-c\mu t} + \int_0^t e^{-c\mu(t-s)} (1+s)^{-\frac{5}{2}} (\delta_0 + \delta_0 Q(t) + Q(t)^2)^2 ds \\
& \leq C(1+t)^{-\frac{5}{2}} (\delta_0 + \delta_0 Q(t) + Q(t)^2)^2.
\end{aligned} \tag{4.68}$$

By summing (4.64), (4.65), (4.66), (4.67) and (4.68), we have

$$Q(t) \leq C\delta_0 + C\delta_0 Q(t) + CQ(t)^2,$$

which proves (4.60) for $\delta_0 > 0$ small enough. This completes the proof of the theorem. \square

Proof of Theorem 2.3. Firstly, (2.46)–(2.49) and (2.42) follow from Theorem 4.6 and Theorem 4.7. Since

$$f_+ = \frac{1}{2}(f_1 + f_2), \quad f_- = \frac{1}{2}(f_1 - f_2), \tag{4.69}$$

this and Theorem 4.7 imply (2.41)–(2.45). \square

Proof of Theorem 2.4. By Eq. (4.58), Theorem 4.6 and Theorem 4.7, we can establish the lower bounds of the time decay rates of macroscopic density, momentum and energy of the global solution (f_1, f_2) to the bVPB system (2.14)–(2.17) and its microscopic part for $t > 0$ large enough. Indeed, it holds for $k = 0, 1$ that

$$\begin{aligned}\|\nabla_x^k(f_1(t), \chi_j)\|_{L_x^2} &\geq \|\nabla_x^k(e^{tE}f_{1,0}, \chi_j)\|_{L_x^2} - \int_0^t \|\nabla_x^k(e^{(t-s)E}G_1(s), \chi_j)\|_{L_x^2} ds \\ &\geq C_1\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}} - C_2\delta_0^2(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \\ \|\nabla_x^k P_1 f_1(t)\|_{L_{x,v}^2} &\geq \|\nabla_x^k P_1(e^{tE}f_{1,0})\|_{L_{x,v}^2} - \int_0^t \|\nabla_x^k P_1(e^{(t-s)E}G_1(s))\|_{L_{x,v}^2} ds \\ &\geq C_1\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}} - C_2\delta_0^2(1+t)^{-\frac{5}{4}-\frac{k}{2}},\end{aligned}$$

which give rise to

$$\begin{aligned}\|f_1(t)\|_{H_w^N} &\geq \|P_0 f_1(t)\|_{L_{x,v}^2} - \|w P_1 f_1(t)\|_{L_{x,v}^2} - \sum_{1 \leq |\alpha| \leq N} \|w \partial_x^\alpha f_1(t)\|_{L_{x,v}^2} \\ &\geq C_1\delta_0(1+t)^{-3/4} - C_2\delta_0^2(1+t)^{-3/4} - C_3\delta_0(1+t)^{-5/4}.\end{aligned}$$

Therefore, for $\delta_0 > 0$ small and $t > 0$ large, we obtain (2.54)–(2.56). By Theorem 4.6, Theorem 4.7 and (4.69), we can prove (2.50)–(2.53). \square

5 The nonlinear problem for mVPB system

5.1 Energy estimates

Let N be a positive integer, and let

$$\begin{aligned}E_k(f) &= \sum_{|\alpha|+|\beta| \leq N} \|w^k \partial_x^\alpha \partial_v^\beta f\|_{L_{x,v}^2}^2 + \sum_{|\alpha| \leq N} \|\partial_x^\alpha \Phi\|_{H_x^1}^2, \\ H_k(f) &= \sum_{|\alpha|+|\beta| \leq N} \|w^k \partial_x^\alpha \partial_v^\beta P_1 f\|_{L_{x,v}^2}^2 + \sum_{|\alpha| \leq N-1} (\|\partial_x^\alpha \nabla_x P_0 f\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{H_x^1}^2), \\ D_k(f) &= \sum_{|\alpha|+|\beta| \leq N} \|w^{\frac{1}{2}+k} \partial_x^\alpha \partial_v^\beta P_1 f\|_{L_{x,v}^2}^2 + \sum_{|\alpha| \leq N-1} (\|\partial_x^\alpha \nabla_x P_0 f\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{H_x^1}^2),\end{aligned}$$

for $k \geq 0$. For brevity, we write $E(f) = E_0(f)$, $H(f) = H_0(f)$ and $D(f) = D_0(f)$ for $k = 0$.

Applying the similar arguments as to derive the equation (4.7)–(4.9) and making use of the system (2.58)–(2.59), we can obtain the compressible Navier-Stokes-Poisson (NSP) equations with inhomogeneous terms for the macroscopic density, momentum and energy $(n, m, q) =: ((f, \chi_0), (f, v\chi_0), (f, \chi_4))$ as follows

$$\partial_t n + \operatorname{div}_x m = 0, \quad (5.1)$$

$$\partial_t m + \partial_t R_6 + \nabla_x n + \sqrt{\frac{2}{3}} \nabla_x q - \nabla_x \Phi = \kappa_5 (\Delta_x m + \frac{1}{3} \nabla_x \operatorname{div}_x m) + n \nabla_x \Phi + R_7, \quad (5.2)$$

$$\partial_t q + \partial_t R_8 + \sqrt{\frac{2}{3}} \operatorname{div}_x m = \kappa_6 \Delta_x q + \sqrt{\frac{2}{3}} \nabla_x \Phi \cdot m + R_9, \quad (5.3)$$

where $\kappa_5, \kappa_6 > 0$ are the viscosity coefficients and the remainder terms R_6, R_7, R_8, R_9 are defined by

$$\begin{aligned}\kappa_5 &= -(L^{-1}P_1(v_1\chi_2), v_1\chi_2), \quad \kappa_6 = -(L^{-1}P_1(v_1\chi_4), v_1\chi_4), \\ R_6 &= (v \cdot \nabla_x L^{-1}P_1 f, v\sqrt{M}), \quad R_7 = -(v \cdot \nabla_x L^{-1}(P_1(v \cdot \nabla_x P_1 f) - P_1 G), v\sqrt{M}), \\ R_8 &= (v \cdot \nabla_x L^{-1}P_1 f, \chi_4), \quad R_9 = -(v \cdot \nabla_x L^{-1}(P_1(v \cdot \nabla_x P_1 f) - P_1 G), \chi_4).\end{aligned}$$

We have the following estimates.

Lemma 5.1 (Macroscopic dissipation). *Let (n, m, q) be a strong solution to (5.1)–(5.3) and assume that the energy $E(f) > 0$ is small enough. Then, there is a constant $p_0 > 0$ so that it holds for $t > 0$ that*

$$\begin{aligned} & \frac{d}{dt} \sum_{k \leq |\alpha| \leq N-1} p_0 (\|\partial_x^\alpha(n, m, q)\|_{L_x^2}^2 + \|\partial_x^\alpha \Phi\|_{H_x^1}^2 + 2 \int_{\mathbb{R}^3} \partial_x^\alpha R_6 \partial_x^\alpha m dx + 2 \int_{\mathbb{R}^3} \partial_x^\alpha R_8 \partial_x^\alpha q dx) \\ & + \frac{d}{dt} \sum_{k \leq |\alpha| \leq N-1} 4 \int_{\mathbb{R}^3} \partial_x^\alpha m \partial_x^\alpha \nabla_x n dx + \sum_{k \leq |\alpha| \leq N-1} (\|\partial_x^\alpha \nabla_x(n, m, q)\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{H_x^1}^2) \\ & \leq C \sqrt{E(f)D(f)} + C \sum_{k \leq |\alpha| \leq N-1} \|\partial_x^\alpha \nabla_x P_1 f\|_{L_{x,v}^2}^2 \end{aligned} \quad (5.4)$$

with $0 \leq k \leq N-1$.

Proof. Taking the inner product between $\partial_x^\alpha m$ and ∂_x^α (5.2) with $|\alpha| \leq N-1$, we have after a direct computation

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\partial_x^\alpha m\|_{L_x^2}^2 + \|\partial_x^\alpha n\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2 + \|\partial_x^\alpha \Phi\|_{L_x^2}^2) + \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha R_6 \partial_x^\alpha m dx \\ & + \sqrt{\frac{2}{3}} \int_{\mathbb{R}^3} \partial_x^\alpha \nabla_x q \partial_x^\alpha m dx + \kappa_5 (\|\partial_x^\alpha \nabla_x m\|_{L_x^2}^2 + \frac{1}{3} \|\partial_x^\alpha \operatorname{div}_x m\|_{L_x^2}^2) \\ & = \int_{\mathbb{R}^3} \partial_x^\alpha (n \nabla_x \Phi) \partial_x^\alpha m dx + \int_{\mathbb{R}^3} \partial_x^\alpha R_7 \partial_x^\alpha m dx + \int_{\mathbb{R}^3} \partial_x^\alpha R_6 \partial_x^\alpha \partial_t m dx - \int_{\mathbb{R}^3} \partial_x^\alpha \Phi \partial_x^\alpha [(e^{-\Phi} - 1) \partial_t \Phi] dx. \end{aligned} \quad (5.5)$$

The first term in the right hand side of (5.5) are bounded by $C \sqrt{E(f)D(f)}$. The second and third terms can be estimated by

$$\begin{aligned} \int_{\mathbb{R}^3} \partial_x^\alpha R_6 \partial_x^\alpha \partial_t m dx & \leq \frac{C}{\epsilon} \|\partial_x^\alpha \nabla_x P_1 f\|_{L_{x,v}^2}^2 + C \sqrt{E(f)D(f)} \\ & + \epsilon (\|\partial_x^\alpha \nabla_x n\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x q\|_{L_x^2}^2), \end{aligned} \quad (5.6)$$

$$\begin{aligned} \int_{\mathbb{R}^3} \partial_x^\alpha R_7 \partial_x^\alpha m dx & \leq C \|\partial_x^\alpha \nabla_x P_1 f\|_{L_{x,v}^2} \|\partial_x^\alpha \nabla_x m\|_{L_x^2} \\ & + C (\|\partial_x^\alpha (\nabla_x \Phi f)\|_{L_{x,v}^2} + \|w^{-\frac{1}{2}} \partial_x^\alpha \Gamma(f, f)\|_{L_{x,v}^2}) \|\partial_x^\alpha \nabla_x m\|_{L_x^2} \\ & \leq \frac{\kappa_5}{2} \|\partial_x^\alpha \nabla_x m\|_{L_x^2}^2 + C \|\partial_x^\alpha \nabla_x P_1 f\|_{L_{x,v}^2}^2 + C \sqrt{E(f)D(f)}, \end{aligned} \quad (5.7)$$

where we have used Lemma 4.1 to obtain

$$\|w^{-\frac{1}{2}} \partial_x^\alpha \Gamma(f, f)\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha (\nabla_x \Phi f)\|_{L_{x,v}^2}^2 \leq C E(f) D(f).$$

For the last term, we make use of (2.59) to obtain

$$\partial_t \Phi - \Delta_x \partial_t \Phi = -\partial_t n - (e^{-\Phi} - 1) \partial_t \Phi = \operatorname{div}_x m - (e^{-\Phi} - 1) \partial_t \Phi, \quad (5.8)$$

which after taking the inner product $\partial_x^\alpha \partial_t \Phi$ and ∂_x^α (5.8) leads for $E(f)$ small enough to

$$\sum_{|\alpha| \leq N-1} (\|\partial_x^\alpha \nabla_x \partial_t \Phi\|_{L_x^2}^2 + \|\partial_x^\alpha \partial_t \Phi\|_{L_x^2}^2) \leq C \sum_{|\alpha| \leq N-1} \|\partial_x^\alpha \operatorname{div}_x m\|_{L_x^2}^2, \quad (5.9)$$

and

$$\int_{\mathbb{R}^3} \partial_x^\alpha \Phi \partial_x^\alpha [(e^{-\Phi} - 1) \partial_t \Phi] dx \leq C \|\nabla_x \Phi\|_{H_x^N} \|\Phi\|_{H_x^N} \|\nabla_x m\|_{H_x^{N-1}} \leq C \sqrt{E(f)D(f)}. \quad (5.10)$$

Therefore, it follows from (5.5), (5.6), (5.7) and (5.10) that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\partial_x^\alpha m\|_{L_x^2}^2 + \|\partial_x^\alpha n\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2 + \|\partial_x^\alpha \Phi\|_{L_x^2}^2) + \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha R_6 \partial_x^\alpha m dx \\ & + \sqrt{\frac{2}{3}} \int_{\mathbb{R}^3} \partial_x^\alpha \nabla_x q \partial_x^\alpha m dx + \frac{\kappa_5}{2} (\|\partial_x^\alpha \nabla_x m\|_{L_x^2}^2 + \frac{1}{3} \|\partial_x^\alpha \operatorname{div}_x m\|_{L_x^2}^2) \end{aligned}$$

$$\leq C\sqrt{E(f)}D(f) + C\|\partial_x^\alpha \nabla_x P_1 f\|_{L_{x,v}^2}^2 + \epsilon(\|\partial_x^\alpha \nabla_x n\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x q\|_{L_x^2}^2). \quad (5.11)$$

Similarly, taking the inner product between $\partial_x^\alpha q$ and $\partial_x^\alpha (5.3)$ with $|\alpha| \leq N-1$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_x^\alpha q\|_{L_x^2}^2 + \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha R_8 \partial_x^\alpha q dx + \sqrt{\frac{2}{3}} \int_{\mathbb{R}^3} \partial_x^\alpha \operatorname{div}_x m \partial_x^\alpha q dx + \frac{1}{2} \kappa_6 \|\partial_x^\alpha \nabla_x q\|_{L_x^2}^2 \\ & \leq C\sqrt{E(f)}D(f) + C\|w^{\frac{1}{2}} \partial_x^\alpha \nabla_x P_1 f\|_{L_{x,v}^2}^2 + \epsilon \|\partial_x^\alpha \nabla_x m\|_{L_x^2}^2. \end{aligned} \quad (5.12)$$

Again, taking the inner product between $\partial_x^\alpha \nabla_x n$ and $\partial_x^\alpha (4.2)$ with $|\alpha| \leq N-1$ to get

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha m \partial_x^\alpha \nabla_x n dx + \frac{1}{2} \|\partial_x^\alpha \nabla_x n\|_{L_x^2}^2 + \|\partial_x^\alpha \Delta_x \Phi\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2 \\ & \leq C\sqrt{E(f)}D(f) + \|\partial_x^\alpha \operatorname{div}_x m\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x q\|_{L_x^2}^2 + C\|\partial_x^\alpha \nabla_x (P_1 f)\|_{L_{x,v}^2}^2. \end{aligned} \quad (5.13)$$

Making the summation $p_0 \sum_{k \leq |\alpha| \leq N-1} [(5.11) + (5.12)] + 4 \sum_{k \leq |\alpha| \leq N-1} (5.13)$ with the constant $p_0 > 0$ large enough and $\epsilon > 0$ small enough, we can obtain (5.4). The proof of the lemma is completed. \square

In the followings, we shall estimate the microscopic part $P_1 f$ appearing in (5.4) in order to enclose the energy estimates of the solution f to mVPB (2.58)–(2.59).

Lemma 5.2 (Microscopic dissipation). *Let (f, Φ) be a strong solution to mVPB (2.58)–(2.59). Then, there are constants $p_k > 0$, $1 \leq k \leq N$ so that it holds for $t > 0$ that*

$$\frac{1}{2} \frac{d}{dt} \sum_{1 \leq |\alpha| \leq N} (\|\partial_x^\alpha f\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \Phi\|_{H_x^1}^2) + \mu \sum_{1 \leq |\alpha| \leq N} \|w^{\frac{1}{2}} \partial_x^\alpha P_1 f\|_{L_{x,v}^2}^2 \leq C\sqrt{E(f)}D(f), \quad (5.14)$$

$$\frac{d}{dt} \|P_1 f\|_{L_{x,v}^2}^2 + \mu \|w^{\frac{1}{2}} P_1 f\|_{L_{x,v}^2}^2 \leq C\|\nabla_x P_0 f\|_{L_{x,v}^2}^2 + CE(f)D(f), \quad (5.15)$$

$$\begin{aligned} & \frac{d}{dt} \sum_{1 \leq k \leq N} p_k \sum_{\substack{|\beta|=k \\ |\alpha|+|\beta| \leq N}} \|\partial_x^\alpha \partial_v^\beta P_1 f\|_{L_{x,v}^2}^2 + \mu \sum_{1 \leq k \leq N} p_k \sum_{\substack{|\beta|=k \\ |\alpha|+|\beta| \leq N}} \|w^{\frac{1}{2}} \partial_x^\alpha \partial_v^\beta P_1 f\|_{L_{x,v}^2}^2 \\ & \leq C \sum_{|\alpha| \leq N-1} (\|\partial_x^\alpha \nabla_x P_0 f\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x P_1 f\|_{L_{x,v}^2}^2) + C\sqrt{E(f)}D(f). \end{aligned} \quad (5.16)$$

Proof. Taking the inner product between $\partial_x^\alpha f$ and $\partial_x^\alpha (2.58)$ with $1 \leq |\alpha| \leq N$ ($N \geq 4$), we have after a tedious computation that

$$\frac{1}{2} \frac{d}{dt} \|\partial_x^\alpha f\|_{L_{x,v}^2}^2 + \frac{1}{2} \frac{d}{dt} (\|\partial_x^\alpha \nabla_x \Phi\|_{L_x^2}^2 + \|\partial_x^\alpha \Phi\|_{L_x^2}^2) + \mu \sum_{1 \leq |\alpha| \leq N} \|w^{\frac{1}{2}} \partial_x^\alpha P_1 f\|_{L_{x,v}^2}^2 \leq C\sqrt{E(f)}D(f). \quad (5.17)$$

In order to enclose the energy estimates, we need to estimate the terms $\partial_x^\alpha \nabla_v f$ with $|\alpha| \leq N-1$. To this end, we rewrite (2.58) as

$$\begin{aligned} & \partial_t (P_1 f) + v \cdot \nabla_x P_1 f + \nabla_x \Phi \cdot \nabla_v P_1 f - L(P_1 f) \\ & = \Gamma(f, f) + \frac{1}{2} v \cdot \nabla_x \Phi P_1 f + P_0 (v \cdot \nabla_x P_1 f + \nabla_x \Phi \cdot \nabla_v P_1 f - \frac{1}{2} v \cdot \nabla_x \Phi P_1 f) \\ & \quad + P_1 (\frac{1}{2} v \cdot \nabla_x \Phi P_0 f - v \cdot \nabla_x P_0 f - \nabla_x \Phi \cdot \nabla_v P_0 f). \end{aligned} \quad (5.18)$$

Taking the inner product between $P_1 f$ and (5.18) and using Cauchy-Schwarz inequality, we can obtain (5.15).

Let $1 \leq k \leq N$, and choose α, β with $|\beta| = k$ and $|\alpha| + |\beta| \leq N$. Taking the inner product between $\partial_x^\alpha \partial_v^\beta P_1 f$ and $\partial_x^\alpha \partial_v^\beta (5.18)$ and summing the resulted equations, we obtain after a tedious computation

$$\sum_{\substack{|\beta|=k \\ |\alpha|+|\beta| \leq N}} \frac{d}{dt} \|\partial_x^\alpha \partial_v^\beta P_1 f\|_{L_{x,v}^2}^2 + \mu \sum_{\substack{|\beta|=k \\ |\alpha|+|\beta| \leq N}} \|w^{\frac{1}{2}} \partial_x^\alpha \partial_v^\beta P_1 f\|_{L_{x,v}^2}^2$$

$$\begin{aligned}
&\leq C \sum_{|\alpha| \leq N-k} (\|\partial_x^\alpha \nabla_x P_0 f\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x P_1 f\|_{L_{x,v}^2}^2) + C_k \sum_{\substack{|\beta| \leq k-1 \\ |\alpha| + |\beta| \leq N}} \|\partial_x^\alpha \partial_v^\beta P_1 f\|_{L_{x,v}^2}^2 \\
&\quad + C\sqrt{E(f)}D(f).
\end{aligned} \tag{5.19}$$

Finally, taking the summation $\sum_{1 \leq k \leq N} p_k(5.19)$ with

$$\mu p_k \geq 2 \sum_{1 \leq j \leq N-k} p_{k+j} C_{k+j}, \quad 1 \leq k \leq N-1, \quad p_N = 1,$$

we obtain (5.16). The proof is completed. \square

With the help of Lemmas 5.1–5.2 we have the following global existence result.

Proposition 5.3. *Let $N \geq 4$. Then, there are equivalent energy functionals $E_0^f(\cdot) \sim E(\cdot)$, $H_0^f(\cdot) \sim H(\cdot)$ and $H_1^f(\cdot) \sim H_1(\cdot)$ so that if the initial energy $E(f_0)$ is sufficiently small, then the Cauchy problem (2.58)–(2.60) of the mVPB system admits a unique global solution $f(x, v, t)$ satisfying*

$$\frac{d}{dt} E_0^f(f(t)) + \mu D(f(t)) \leq 0, \tag{5.20}$$

$$\frac{d}{dt} H_0^f(f(t)) + \mu D(f(t)) \leq C \|\nabla_x P_0 f(t)\|_{L_{x,v}^2}^2, \tag{5.21}$$

$$\frac{d}{dt} H_1^f(f(t)) + \mu D_1(f(t)) \leq C \|\nabla_x P_0 f(t)\|_{L_{x,v}^2}^2. \tag{5.22}$$

5.2 Convergence rates

With the help of the energy estimates established in Sect. 5.1, we are able to show Theorems 2.7–2.8 for the Cauchy problem of the nonlinear mVPB system (2.58)–(2.60) in this subsection.

Proof of Theorem 2.7. Let f be the global solution to the IVP problem (2.58)–(2.60) for $t > 0$. We can represent it in terms of the semigroup e^{tB_m} as

$$f(t) = e^{tB_m} f_0 + \int_0^t e^{(t-s)B_m} G(s) ds + \int_0^t e^{(t-s)B_m} \operatorname{div}_x (I - \Delta_x)^{-1} V(s) ds, \tag{5.23}$$

where the nonlinear terms G and V are defined by

$$G = \frac{1}{2}(v \cdot \nabla_x \Phi) f - \nabla_x \Phi \cdot \nabla_v f + \Gamma(f, f), \quad V = v\sqrt{M}(e^\Phi + \Phi - 1).$$

Define a functional $Q_1(t)$ for the global solution f to the IVP problem (2.58)–(2.60) for any $t > 0$ as

$$\begin{aligned}
Q_1(t) = \sup_{0 \leq s \leq t} \sum_{k=0}^1 \Big\{ & \left(\sum_{j=0}^4 \|\partial_x^k(f(t), \chi_j)\|_{L_x^2} + \|\partial_x^k \Phi(s)\|_{H_x^1} \right) (1+s)^{\frac{3}{4} + \frac{k}{2}} + \|\partial_x^k P_1 f(s)\| (1+s)^{\frac{5}{4} + \frac{k}{2}} \\
& + (\|P_1 f(s)\|_{H_w^N} + \|\nabla_x P_0 f(s)\|_{L_v^2(H_x^{N-1})}) (1+s)^{\frac{5}{4}} \Big\}.
\end{aligned}$$

We shall show below that it holds under the assumptions of Theorem 2.7 that

$$Q_1(t) \leq C\delta_0. \tag{5.24}$$

To this end, we first deal with the time-decay rates of the macroscopic density, momentum and energy, which in terms of (5.23) satisfy the following equations

$$(f(t), \chi_j) = (e^{tB_m} f_0, \chi_j) + \int_0^t (e^{(t-s)B_m} G(s), \chi_j) ds + \int_0^t (e^{(t-s)B_m} \operatorname{div}_x (I - \Delta_x)^{-1} V(s), \chi_j) ds.$$

By (3.84)–(3.86), we have for any $\alpha, \alpha' \in \mathbb{N}^3$ with $\alpha' \leq \alpha$ that

$$\|\partial_x^\alpha (e^{tB_m} \partial_x (I - \Delta_x)^{-1} f_0, \chi_j)\|_{L_x^2} \leq C(1+t)^{-\frac{5}{4}-\frac{k}{2}} (\|\partial_x^\alpha f_0\|_{L_{x,v}^2} + \|\partial_x^{\alpha'} f_0\|_{L^{2,1}}), \quad j = 0, 1, 2, 3, 4, \quad (5.25)$$

$$\|\partial_x^\alpha (I - \Delta_x)^{-1} (e^{tB_m} \partial_x (I - \Delta_x)^{-1} f_0, \sqrt{M})\|_{H_x^1} \leq C(1+t)^{-\frac{5}{4}-\frac{k}{2}} (\|\partial_x^\alpha f_0\|_{L_{x,v}^2} + \|\partial_x^{\alpha'} f_0\|_{L^{2,1}}), \quad (5.26)$$

$$\|\partial_x^\alpha P_1(e^{tB_m} \partial_x (I - \Delta_x)^{-1} f_0)\|_{L_{x,v}^2} \leq C(1+t)^{-\frac{7}{4}-\frac{k}{2}} (\|\partial_x^\alpha f_0\|_{L_{x,v}^2} + \|\partial_x^{\alpha'} f_0\|_{L^{2,1}}), \quad (5.27)$$

with $k = |\alpha - \alpha'|$. Then, we can estimate the nonlinear terms $G(s)$ and $V(s)$ for $0 \leq s \leq t$ as below

$$\|G(s)\|_{L_{x,v}^2} \leq C\|wf\|_{L^{2,3}}\|f\|_{L^{2,6}} + \|\nabla_x \Phi\|_{L_x^3} (\|wf\|_{L^{2,6}} + \|\nabla_v f\|_{L^{2,6}}) \leq C(1+s)^{-2} Q_1(t)^2, \quad (5.28)$$

$$\|G(s)\|_{L^{2,1}} \leq C\|f\|_{L_{x,v}^2}\|wf\|_{L_{x,v}^2} + \|\nabla_x \Phi\|_{L_x^2} (\|wf\|_{L_{x,v}^2} + \|\nabla_v f\|_{L_{x,v}^2}) \leq C(1+s)^{-\frac{3}{2}} Q_1(t)^2, \quad (5.29)$$

$$\|V(s)\|_{L_{x,v}^2} = \sqrt{3}\|e^{-\Phi} + \Phi - 1\|_{L_x^2} \leq Ce^{\|\Phi\|_{L_x^\infty}} \|\Phi\|_{L_x^3} \|\Phi\|_{L_x^6} \leq C(1+s)^{-2} e^{CQ_1(t)} Q_1(t)^2, \quad (5.30)$$

$$\|V(s)\|_{L^{2,1}} = \sqrt{3}\|e^{-\Phi} + \Phi - 1\|_{L_x^1} \leq Ce^{\|\Phi\|_{L_x^\infty}} \|\Phi\|_{L_x^2} \|\Phi\|_{L_x^2} \leq C(1+s)^{-\frac{3}{2}} e^{CQ_1(t)} Q_1(t)^2. \quad (5.31)$$

It follows from (2.65), (5.25) and (5.28)–(5.31) that

$$\begin{aligned} \|(f(t), \chi_j)\|_{L_x^2} &\leq C(1+t)^{-\frac{3}{4}} (\|f_0\|_{L_{x,v}^2} + \|f_0\|_{L^{2,1}}) \\ &\quad + C \int_0^t (1+t-s)^{-\frac{3}{4}} (\|G(s)\|_{L_{x,v}^2} + \|G(s)\|_{L^{2,1}}) ds \\ &\quad + C \int_0^t (1+t-s)^{-\frac{5}{4}} (\|V(s)\|_{L_{x,v}^2} + \|V(s)\|_{L^{2,1}}) ds \\ &\leq C\delta_0(1+t)^{-\frac{3}{4}} + C(1+t)^{-\frac{3}{4}} e^{CQ_1(t)} Q_1(t)^2. \end{aligned} \quad (5.32)$$

Similarly, we have

$$\begin{aligned} \|\nabla_x(f(t), \chi_j)\|_{L_x^2} &\leq C(1+t)^{-\frac{5}{4}} (\|\nabla_x f_0\|_{L_{x,v}^2} + \|f_0\|_{L^{2,1}}) \\ &\quad + C \int_0^t (1+t-s)^{-\frac{5}{4}} (\|\nabla_x G(s)\|_{L_{x,v}^2} + \|G(s)\|_{L^{2,1}}) ds \\ &\quad + C \int_0^t (1+t-s)^{-\frac{7}{4}} (\|\nabla_x V(s)\|_{L_x^2} + \|V(s)\|_{L^{2,1}}) ds \\ &\leq C\delta_0(1+t)^{-\frac{5}{4}} + C(1+t)^{-\frac{5}{4}} e^{CQ_1(t)} Q_1(t)^2, \end{aligned} \quad (5.33)$$

where we have used

$$\begin{aligned} \|\nabla_x G(s)\|_{L_{x,v}^2} + \|G(s)\|_{L^{2,1}} &\leq C(1+s)^{-\frac{3}{2}} Q_1(t)^2, \\ \|\nabla_x V(s)\|_{L_{x,v}^2} + \|V(s)\|_{L^{2,1}} &\leq C(1+s)^{-\frac{3}{2}} e^{CQ_1(t)} Q_1(t)^2. \end{aligned}$$

By (2.59), we obtain

$$\Phi = -(I - \Delta_x)^{-1} (f, \sqrt{M}) + (I - \Delta_x)^{-1} (e^{-\Phi} + \Phi - 1),$$

which implies that

$$\begin{aligned} \Phi(t) &= -(I - \Delta_x)^{-1} (e^{tB_m} f_0, \sqrt{M}) - \int_0^t (I - \Delta_x)^{-1} (e^{(t-s)B_m} G(s), \sqrt{M}) ds \\ &\quad + \int_0^t (I - \Delta_x)^{-1} (e^{(t-s)B_m} \operatorname{div}_x (I - \Delta_x)^{-1} V(s), \sqrt{M}) ds + (I - \Delta_x)^{-1} (e^{-\Phi} + \Phi - 1). \end{aligned} \quad (5.34)$$

Due to the fact

$$\|\nabla_x^k (I - \Delta_x)^{-1} (e^{-\Phi} + \Phi - 1)\|_{H_x^1} \leq \|e^{-\Phi} + \Phi - 1\|_{L_x^2} \leq C(1+t)^{-2} e^{CQ_1(t)} Q^2(t), \quad k = 0, 1, \quad (5.35)$$

we obtain by (2.66) and (5.26) that

$$\|\Phi(t)\|_{H_x^1} \leq C\delta_0(1+t)^{-\frac{3}{4}} + C(1+t)^{-\frac{3}{4}}e^{CQ_1(t)}Q^2(t) + C(1+t)^{-2}e^{CQ_1(t)}Q^2(t), \quad (5.36)$$

$$\|\nabla_x \Phi(t)\|_{H_x^1} \leq C\delta_0(1+t)^{-\frac{5}{4}} + C(1+t)^{-\frac{5}{4}}e^{CQ_1(t)}Q^2(t) + C(1+t)^{-2}e^{CQ_1(t)}Q^2(t). \quad (5.37)$$

Next, we estimate the microscopic part $P_1 f(t)$. Since $P_1 f(t)$ satisfies

$$P_1 f(t) = P_1(e^{tB_m} f_0) + \int_0^t P_1(e^{(t-s)B_m} G(s))ds + \int_0^t P_1(e^{(t-s)B_m} \operatorname{div}_x(I - \Delta_x)^{-1} V(s))ds,$$

it follows from (2.67) and (5.27) that

$$\begin{aligned} \|P_1 f(t)\|_{L_{x,v}^2} &\leq C(1+t)^{-\frac{5}{4}}(\|f_0\|_{L_{x,v}^2} + \|f_0\|_{L^{2,1}}) \\ &\quad + C \int_0^t (1+t-s)^{-\frac{5}{4}}(\|G(s)\|_{L_{x,v}^2} + \|G(s)\|_{L^{2,1}})ds \\ &\quad + C \int_0^t (1+t-s)^{-\frac{7}{4}}(\|V(s)\|_{L_{x,v}^2} + \|V(s)\|_{L^{2,1}})ds \\ &\leq C\delta_0(1+t)^{-\frac{5}{4}} + C(1+t)^{-\frac{5}{4}}e^{CQ_1(t)}Q_1(t)^2, \end{aligned} \quad (5.38)$$

and

$$\begin{aligned} \|\nabla_x P_1 f(t)\|_{L_{x,v}^2} &\leq C(1+t)^{-\frac{7}{4}}(\|\nabla_x f_0\|_{L_{x,v}^2} + \|f_0\|_{L^{2,1}}) \\ &\quad + C \int_0^{t/2} (1+t-s)^{-\frac{7}{4}}(\|\nabla_x G(s)\|_{L_{x,v}^2} + \|G(s)\|_{L^{2,1}})ds \\ &\quad + C \int_{t/2}^t (1+t-s)^{-\frac{5}{4}}(\|\nabla_x G(s)\|_{L_{x,v}^2} + \|\nabla_x G(s)\|_{L^{2,1}})ds \\ &\quad + C \int_0^t (1+t-s)^{-\frac{7}{4}}(\|\nabla_x V(s)\|_{L_{x,v}^2} + \|\nabla_x V(s)\|_{L^{2,1}})ds \\ &\leq C\delta_0(1+t)^{-\frac{7}{4}} + C(1+t)^{-\frac{7}{4}}e^{CQ_1(t)}Q_1(t)^2, \end{aligned} \quad (5.39)$$

due to the facts

$$\begin{aligned} \|\nabla_x G(s)\|_{L_{x,v}^2} + \|\nabla_x G(s)\|_{L^{2,1}} &\leq C(1+s)^{-2}Q_1(t)^2, \\ \|\nabla_x V(s)\|_{L_{x,v}^2} + \|\nabla_x V(s)\|_{L^{2,1}} &\leq C(1+s)^{-2}e^{CQ_1(t)}Q_1(t)^2. \end{aligned}$$

Finally, the higher order estimates can be established in terms of (5.22) and $d_1 H_1^f(f) \leq D_1(f)$ for some constant $d_1 > 0$ as

$$\begin{aligned} H_1^f(f(t)) &\leq e^{-d_1 \mu t} H_1^f(f_0) + \int_0^t e^{-d_1 \mu(t-s)} \|\nabla_x P_0 f(s)\|_{L_{x,v}^2}^2 ds \\ &\leq C\delta_0^2 e^{-d_1 \mu t} + \int_0^t e^{-d_1 \mu(t-s)} (1+s)^{-\frac{5}{2}} (\delta_0 + Q_1(t)^2)^2 ds \\ &\leq C(1+t)^{-\frac{5}{2}} (\delta_0 + e^{CQ_1(t)} Q_1(t)^2)^2. \end{aligned} \quad (5.40)$$

By summing (5.32), (5.33), (5.36), (5.37), (5.38), (5.39) and (5.40) together, we have

$$Q_1(t) \leq C\delta_0 + Ce^{CQ_1(t)}Q_1(t)^2,$$

which leads to (5.24) for $\delta_0 > 0$ small enough. This completes the proof of the proposition. \square

Proof of Theorem 2.8. By Eq. (5.23) and Theorem 2.7, we can establish the lower bounds of the time decay rates of macroscopic density, momentum and energy of the global solution f and its microscopic part. Indeed, it holds for $t > 0$ large enough and $k = 0, 1$ that

$$\|\nabla_x^k(f(t), \chi_j)\|_{L_x^2} \geq \|\nabla_x^k(e^{tB_m} f_0, \chi_j)\|_{L_x^2} - \int_0^t \|\nabla_x^k(e^{(t-s)B_m} G(s), \chi_j)\|_{L_x^2} ds$$

$$\begin{aligned}
& - \int_0^t \|\nabla_x^k (e^{(t-s)B_m} \operatorname{div}_x (I - \Delta_x)^{-1} V(s), \chi_j)\|_{L_x^2} ds \\
& \geq C_1 \delta_0 (1+t)^{-\frac{3}{4}-\frac{k}{2}} - C_2 \delta_0^2 (1+t)^{-\frac{3}{4}-\frac{k}{2}}, \\
\|\nabla_x^k P_1 f(t)\|_{L_{x,v}^2} & \geq \|\nabla_x^k P_1 (e^{tB_m} f_0)\|_{L_{x,v}^2} - \int_0^t \|\nabla_x^k P_1 (e^{(t-s)B_m} G(s))\|_{L_{x,v}^2} ds \\
& - \int_0^t \|\nabla_x^k P_1 (e^{(t-s)B_m} \operatorname{div}_x (I - \Delta_x)^{-1} V(s))\|_{L_{x,v}^2} ds \\
& \geq C_1 \delta_0 (1+t)^{-\frac{5}{4}-\frac{k}{2}} - C_2 \delta_0^2 (1+t)^{-\frac{5}{4}-\frac{k}{2}},
\end{aligned}$$

and by (5.34) and (5.35) that

$$\begin{aligned}
\|\nabla_x^k \Phi(t)\|_{H_x^1} & \geq \|\nabla_x^k (I - \Delta_x)^{-1} (e^{tB_m} f_0, \sqrt{M})\|_{H_x^1} - \int_0^t \|\nabla_x^k (I - \Delta_x)^{-1} (e^{(t-s)B_m} G(s), \sqrt{M})\|_{H_x^1} ds \\
& - \int_0^t \|\nabla_x^k (I - \Delta_x)^{-1} (e^{(t-s)B_m} \operatorname{div}_x (I - \Delta_x)^{-1} V(s), \sqrt{M})\|_{H_x^1} ds \\
& - \|\nabla_x^k (I - \Delta_x)^{-1} (e^{-\Phi} + \Phi - 1)\|_{H_x^1} \\
& \geq C_1 \delta_0 (1+t)^{-\frac{3}{4}-\frac{k}{2}} - C_2 \delta_0^2 (1+t)^{-\frac{3}{4}-\frac{k}{2}} - C_2 \delta_0^2 (1+t)^{-2}.
\end{aligned} \tag{5.41}$$

These and Theorem 2.7 give rise to

$$\begin{aligned}
\|f(t)\|_{H_w^N} & \geq \|P_0 f(t)\|_{L_{x,v}^2} - \|w P_1 f(t)\|_{L_{x,v}^2} - \sum_{1 \leq |\alpha| \leq N} \|w \partial_x^\alpha f(t)\|_{L_{x,v}^2} \\
& \geq C_1 \delta_0 (1+t)^{-3/4} - C_2 \delta_0^2 (1+t)^{-3/4} - C_3 \delta_0 (1+t)^{-5/4}.
\end{aligned}$$

Therefore, we obtain (2.74)–(2.77) for $\delta_0 > 0$ sufficiently and $t > 0$ large enough. \square

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References

- [1] C. Cercignani, R. Illner and M. Pulvirenti, *The Mathematical Theory of Dilute Gases*, Applied Mathematical Sciences, 106. Springer-Verlag, New York, 1994.
- [2] S. Codier and E. Grenier, Quasineutral limit of an Euler-Poisson system arising from plasma physics. *Commun. Part. Diff. Eq.*, 25 (2000), 1099-1113.
- [3] R.J. Duan and R. M. Strain, Optimal time decay of the Vlasov-Poisson-Boltzmann system in \mathbb{R}^3 , *Arch. Ration. Mech. Anal.*, 199 (2011), no. 1, 291-328.
- [4] R.J. Duan and T. Yang, Stability of the one-species Vlasov-Poisson-Boltzmann system, *SIAM J. Math. Anal.*, 41 (2010), 2353-2387.
- [5] R.J. Duan, T. Yang and C.J. Zhu, Boltzmann equation with external force and Vlasov-Poisson-Boltzmann system in infinite vacuum, *Discrete Contin. Dyn. Syst.*, 16 (2006), 253-277.
- [6] R.S. Ellis and M.A. Pinsky, The first and second fluid approximations to the linearized Boltzmann equation. *J. Math. pure et appl.*, 54 (1975), 125-156.
- [7] Y. Guo, The Vlasov-Poisson-Boltzmann system near Maxwellians, *Comm. Pure Appl. Math.*, 55 (9) (2002), 1104-1135.
- [8] Y. Guo, The Vlasov-Poisson-Boltzmann system near vacuum, *Comm. Math. Phys.*, 218 (2) (2001), 293-313.
- [9] H.L. Li, T. Yang and M.Y. Zhong, Spectral analysis for the Vlasov-Poisson-Boltzmann system. Preprint.
- [10] T.-P. Liu and S.-H. Yu, The Greens function and large-time behavior of solutions for the one-dimensional Boltzmann equation, *Comm. Pure Appl. Math.*, 57 (2004), 1543-1608.

- [11] T.-P. Liu, T. Yang and S.-H. Yu, Energy method for the Boltzmann equation, *Physica D*, 188 (3-4) (2004), 178-192.
- [12] P.A. Markowich, C.A. Ringhofer and C. Schmeiser, *Semiconductor Equations*, Springer-Verlag, Vienna, 1990. x+248 pp.
- [13] S. Mischler, On the initial boundary value problem for the Vlasov-Poisson-Boltzmann system, *Comm. Math. Phys.*, 210 (2000), 447-466.
- [14] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.
- [15] S. Ukai, On the existence of global solutions of mixed problem for non-linear Boltzmann equation, *Proceedings of the Japan Academy*, 50 (1974), 179-184.
- [16] S. Ukai and T. Yang, The Boltzmann equation in the space $L^2 \cap L^\infty_\beta$: Global and time-periodic solutions, *Analysis and Applications*, 4 (2006), 263-310.
- [17] S. Ukai, T. Yang, *Mathematical Theory of Boltzmann Equation*. Lecture Notes Series-No. 8, Hong Kong: Liu Bie Ju Center for Mathematical Sciences, City University of Hong Kong, March 2006.
- [18] T. Yang, H.J. Yu and H.J. Zhao, Cauchy problem for the Vlasov-Poisson-Boltzmann system, *Arch. Rational Mech. Anal.*, 182 (2006), 415-470.
- [19] T. Yang and H.J. Zhao, Global existence of classical solutions to the Vlasov-Poisson- Boltzmann system, *Comm. Math. Phys.*, 268 (2006), 569-605.
- [20] T. Yang, H.J. Yu, Optimal convergence rates of classical solutions for Vlasov-Poisson-Boltzmann system. *Commun. Math. Phys.* 301 (2011), 319-355.
- [21] Alexander Sotirov and Shih-Hsien Yu, On the Solution of a Boltzmann System for Gas Mixtures. *Arch. Rational Mech. Anal.*, 195 (2010) 675-700.
- [22] M. Zhong, Optimal time-decay rate of the Boltzmann equation. *Sci China Math*, 2014, 57: 807-822, doi: 10.1007/s11425-013-4621-1.